Control theory: state-space approach for linear systems

State-feedback control

2 janvier 2022

Principle of state-feedabck control

We are going to study the design of control-laws for systems described by state-space representations

continuous time

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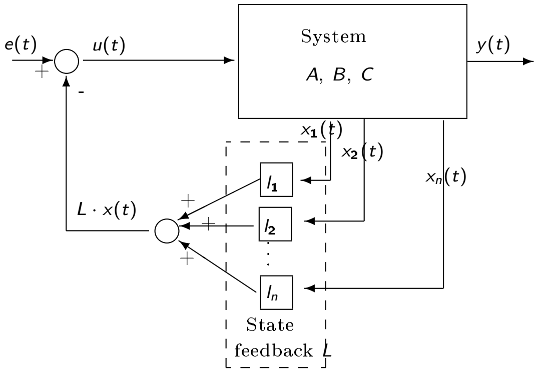
  These variables are assumed to be all measured (we have knowledge of their values at any time-instant).
  In the next chapter, we will see how to *estimate* these values if they cannot be measured.

Principle of state-feedabck control



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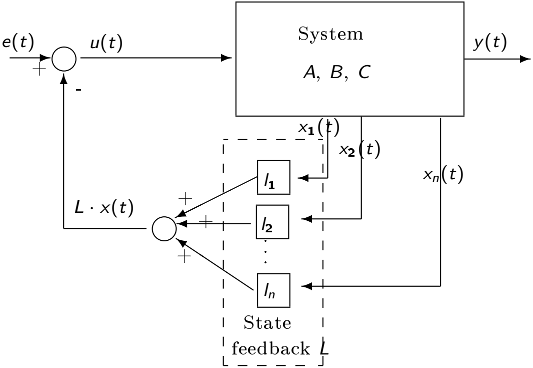

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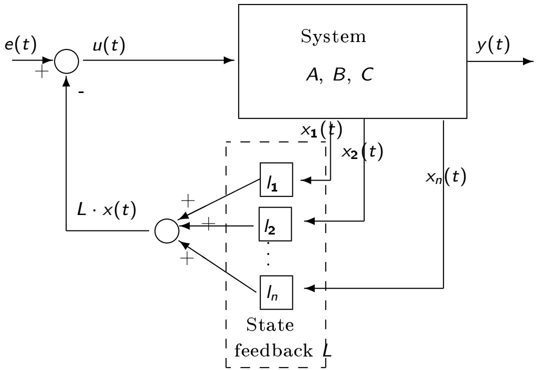

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  Behavior of the system is taken into account through $L \cdot x(t)$.

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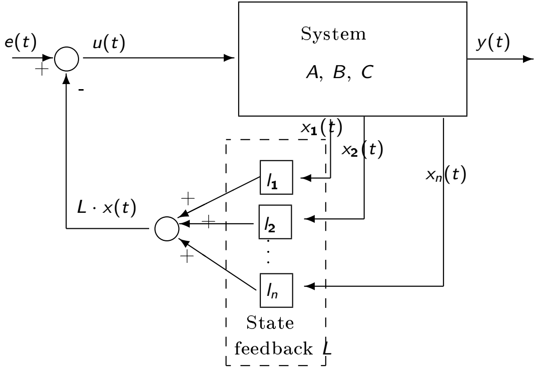

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  Behavior of the system is taken into account through $L \cdot x(t)$.
- L is the *feedback matrix* to be found.

Pole-placement

State-equation with the feedback can be written

$$\dot{x}(t) = Ax(t) + B\left(e(t) - Lx(t)\right) = (A - BL)x(t) + Be(t)$$

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Choosing feedback-matrix L in order to impose the poles for the sytem with the feedback, that is the eigenvalues of $(A - BL)$ or the roots of $P(s) = |s\mathbf{I} - A + BL|$.

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Pole-placement equation

Let $P_{des}(s)$ be a desired characteristic polynomial for the system with the feedback, it is necessary to solve

$$|s\mathbf{I} - A + BL| = P_{des}(s)$$

Pole-placement control design

Software-aided solution

Scilab's function `ppol()` solves pole-placement equation

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Formal solution (for the SISO case)

- Reminder :

$$s^n + \gamma_{n-1} s^{n-1} + \ldots + \gamma_0 = s^n + \delta_{n-1} s^{n-1} + \ldots + \delta_0$$

if, and only if,

$$\gamma_0 = \delta_0,\ \gamma_1 = \delta_1, \ldots,\ \gamma_{n-1} = \delta_{n-1}$$

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- It is then sufficient to identify coefficients of L in :

$$|s\mathbf{I} - A + BL| = P_{des}(s)$$

Choice of the poles

The coefficients chosen into $P_{des}(s)$ transcribe the poles which have been chosen for the system once the feedback is applied. In other words, the feedback makes it possible to impose $P_{des}(s)$ as the denominator of the transfer function for the controlled system.

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First-order case

The denominator of the transfer function is $1 + Ts$, or equivalently $1/T + s$, and the choice for α_0 in polynomial $P_{des}(s) = s + \alpha_0$ imposes a desired value for time-constant T (describing "quickness" of the system).

Choice of the poles

Second-order case

Denominator of the transfer function :$1 + \frac{2\xi}{\omega_n}s + \frac{1}{\omega_n^2}s^2$ or $\omega_n^2 + 2\xi\omega_n s + s^2$.
The choices for α_0 and α_1 into $P_{des}(s) = s^2 + \alpha_1 s + \alpha_0$ impose desired values for ω_n and ξ.

- Feedback has a "physical" meaning only if l_0 and l_1 are some real numbers : $\alpha 0$ and $\alpha 1$ must be chosen accordingly (roots of $P_{des}(s)$ can be conjugate complex numbers).
- To make the system stable, ξ must be strictly positive ($\Leftrightarrow$ roots of $P_{des}(s)$ must all have a strictly negative real part). If under-damped second-order system ($0 < \xi < 1$), then ξ determines the range of overshoot(s).
- For a given value for ξ, the choice of the natural frequency makes it possible to tune the "quickness" of the system (for instance, by imposing the 5% response-time).

Example : container handling gantry crane

Reminder of previous results

State-space representation :

$$\left\{ \begin{array}{rcl} \dot{x}(t) & = & \begin{pmatrix} 0 & g \\ -\frac{1}{L} & 0 \end{pmatrix} x(t) + \begin{pmatrix} 0 \\ \frac{1}{L} \end{pmatrix} u(t) \\ y(t) & = & \begin{pmatrix} 1 & 0 \end{pmatrix} x(t) \end{array} \right. .$$

It has been shown that the system is unstable.
It is a second-order system with

- $\xi = 0$,
- $\omega_n = \sqrt{\frac{g}{L}}$.

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A state-feedback control is designed in order to place the poles of the system.

For instance, one can impose that the controlled system admits for characteristic polynomial :

$$P_{des}(s) = (s + 2)^2$$

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a) Its roots (double root -2) are then all with srictly negative real part, which means that the system is stabilized thanks to the state-feedback.

b) This makes it possible to impose s^2+4s+4, or equivalently $\frac{1}{4}s^2+s+1$ as denominator of the transfer function. This imposes that $\xi = 1$: step response will be without overshoot.

Example : container handling gantry crane

Finding matrix regulation L to obtain $P_{des}(s)$ as characteristic polynomial for the controlled system.

1. We have :

$$s\mathbf{I} - A + BL = \begin{pmatrix} s & -g \\ \frac{1+l_1}{L} & s + \frac{l_2}{L} \end{pmatrix}$$

   hence $|s\mathbf{I} - A + BL| = s^2 + s\frac{l_2}{L} + \frac{g+l_1 g}{L}$.

2. We want

$$\begin{array}{rcl} |s\mathbf{I} - A + BL| & = & P_{des}(s) \\ s^2 + s\frac{l_2}{L} + \frac{g+l_1 g}{L} & = & s^2 + 4s + 4 \end{array}$$

3. Identification of polynomial coefficients leads to

$$\left\{ \begin{array}{rcl} l_1 & = & \frac{4L}{g} - 1 \\ l_2 & = & 4L \end{array} \right.$$

Steady state for the system equipped with the feedback

Steady state

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- Behavior of the system once its state does not evolve anymore.
- Equivalently, behavior of the system when reference input e is constant and when time (t or k) tends to infinity.

Steady state in the continuous-time case

- Let us denote

$$e = \lim_{t \to \infty} e(t), \quad x = \lim_{t \to \infty} x(t), \quad y = \lim_{t \to \infty} y(t)$$

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$$y \;=\; -C(A-BL)^{-1}Be$$

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- $$\left\{ \begin{array}{rcl} x(k) & = & (A - BL)x(k) + Be(k) \\ y(k) & = & Cx(k) \end{array} \right.$$
- This leads to
$$y = C(\mathbf{I} - A + BL)^{-1}Be$$
by denoting y and e the constant signals $y(k)$ and $u(k)$ during the steady state.

To obtain an unitary static gain

Static gain y/e of the system equipped with the feedback

$$-C(A - BL)^{-1}B \quad \text{(continuous-time)} \qquad C(\mathbf{I} - A + BL)^{-1}B \quad \text{(discrete-time)}$$

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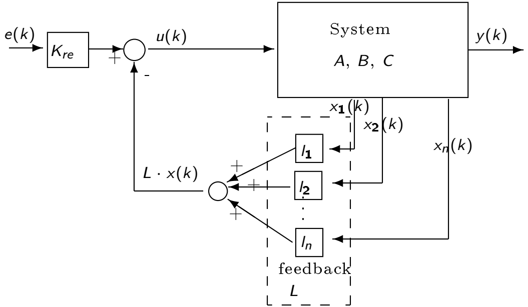
To obtain an unitary gain between e and output y, we can add a gain K_{re} equal to the inverse to this static gain.



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The steady-state gain is $-C(A-BL)^{-1}B$. We have

$$A - BL = \begin{pmatrix} 0 & g \\ -\frac{4}{g} & -4 \end{pmatrix}, \text{ hence } (A-BL)^{-1} = \begin{pmatrix} -1 & -\frac{g}{4} \\ \frac{1}{g} & 0 \end{pmatrix}$$

and

$$-C(A-BL)^{-1}B = \frac{g}{4L}.$$

The additionnal gain is then $K_{re} = \frac{1}{-C(A-BL)^{-1}B} = \frac{4L}{g}$.

Adding an integral effect

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Let's pose $z(t) = \int_0^t (e(t) - y(t))dt \Leftrightarrow \dot{z}(t) = e(t) - y(t)$.

Adding an integral effect (ctd)

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$$\iff \left\{\begin{array}{rcl}\dot{x}(t) & = & (A-BL)x(t)+BL_i z(t)+p\\ \dot{z}(t) & = & e(t)-y(t)\end{array}\right.$$

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$$\Longleftrightarrow \begin{pmatrix} \dot{x}(t) \\ \dot{z}(t) \end{pmatrix} = \begin{pmatrix} A - BL & BL_i \\ -C & 0 \end{pmatrix} \begin{pmatrix} x(t) \\ z(t) \end{pmatrix} + \begin{pmatrix} p \\ e(t) \end{pmatrix}.$$

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The evolution matrix of the looped system is always of the form $A' - B'L'$, and the choice of L and L_i is then also made by pole-placement.