# Control theory: state-space approach for linear systems

## Discrete-time systems

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Linear systems described by state-space representations

$$\left\{ \begin{array}{rcl} x(k+1) & = & Ax(k) + Bu(k) \\ y(k) & = & Cx(k) \end{array} \right. \quad (1)$$

in which the time, denoted $k$, takes some discrete values (e.g. $\mathbb{Z}$).

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Same observation as in the continuous-time case :

- A given system admits several equivalent state-space representations.
- Consequentlty, from a transfer function there exist several methods to get the distinct but equivalent state-space representations of a system, each of them having a particular form.

# From a TF to a state-space repr.

## Controllable Canonical Form

$$\left\{\begin{array}{rcl} x(k+1) & = & \begin{pmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & 1 \\ -a_0 & -a_1 & -a_2 & \dots & -a_{n-1} \end{pmatrix} x(k) + \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix} u(k) \\ y(k) & = & \begin{pmatrix} b_0 & b_1 & \dots & b_{n-1} \end{pmatrix} x(k) \end{array}\right. .$$

# From a TF to a state-space repr.

## Observable Canonical Form

$$\left\{ \begin{array}{rcl} x(k+1) & = & \begin{pmatrix} -a_{n-1} & 1 & 0 & \dots & 0 \\ -a_{n-2} & 0 & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ -a_1 & 0 & 0 & \dots & 1 \\ -a_0 & 0 & 0 & \dots & 0 \end{pmatrix} x(k) + \begin{pmatrix} b_{n-1} \\ b_{n-2} \\ \vdots \\ b_1 \\ b_n \end{pmatrix} u(k) \\ \\ y(k) & = & \begin{pmatrix} 1 & 0 & \dots & 0 \end{pmatrix} x(k) \end{array} \right. .$$

# Solution to state-space equations

### Theorem

*$x(k_0)$ value of the state at initial time-instant $k_0$, state at $k$ is given by*

$$x(k) = A^{k-k_0}x(k_0) + \sum_{l=k_0}^{k-1} A^{k-1-l}Bu(l)\,, \tag{2}$$

*and the output can be written*

$$y(k) = CA^{k-k_0}x(k_0) + \sum_{l=k_0}^{k-1} CA^{k-1-l}Bu(l)\,. \tag{3}$$

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The proof proceeds by induction.

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$$x(k_1) = A^{k_1 - k_0} x(k_0) + \sum_{l=k_0}^{k_1 - 1} A^{k_1 - 1 - l} B u(l) \,.$$

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If $u(k) = 0$ for $k > k_1$, state for $k \geq k_1$ can be written :

$$\begin{aligned} x(k) &= A^{k-k_1}x(k_1) + \sum_{l=k_1}^{k-1} A^{k-1-l}Bu(l) \\ &= A^{k-k_1}x(k_1) \qquad \text{(since } u(l) = 0 \text{ for } l \geq k_1) , \end{aligned}$$

which tends to zero while $k \to \infty$ if $\lim_{k\to\infty} A^{k-k_1} = \mathbf{0}$.

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## Theorem

*A linear system is said to be stable if, and only if, all the eigenvalues of its matrix evolution have a modulus strictly less than 1.*

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The *characteristic polynomial* $P(z)$ of a linear system is defined by

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## Corollary (Stability criterion)

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# Coontrollability

*Analogous to the time-continuous case !*

## Theorem (Controllability criterion)

*A linear system is said to be controllable if, and only if,*

$$\operatorname{rank} \overbrace{\left(B|AB|A^2B|\dots|A^{n-1}B\right)}^{\Gamma_{com}} = n \,, \tag{5}$$

*where $n$ is the dimension of matrix evolution $A$.*

# Observability

*Analogous to the time-continuous case !*

## Theorem (Observability criterion)

*A linear system is said to be observable if, and only if,*

$$\operatorname{rank} \begin{pmatrix} C \\ CA \\ \vdots \\ CA^{n-1} \end{pmatrix} = n. \tag{6}$$

# Continuous-time systems studied by means of a computer : sampled discrete-time systems

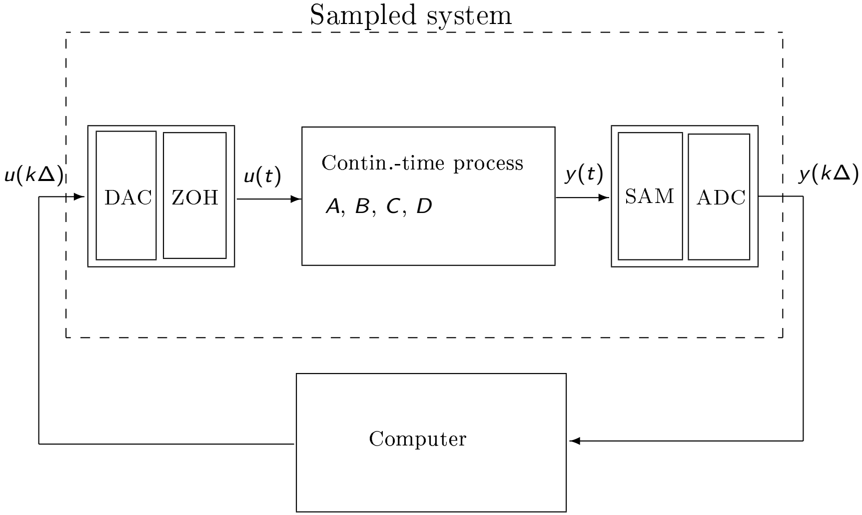


Figure – Continuous-time system seen as a sampled discrete-time system by the

# Sampled systems : discretizing a continuous-time state-equation

- We consider a continuous-time state-space representation :

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- When it is sampled, it can be described at sampling time-instants $k\Delta$, denoted $k$ to lighten equations, by discrete-time equations :

$$\left\{ \begin{array}{rcl} x(k+1) & = & A_{ech}x(k) + B_{ech}u(k) \\ y(k) & = & C_{ech}x(k) \end{array} \right. \quad (8)$$

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- We want to figure out matrices $A_{ech}$, $B_{ech}$ and $C_{ech}$ from known matrices $A$, $B$ and $C$.

# Sampled systems : discretizing a continuous-time state-equation

Knowing the state-value at sampling-time $t_k = k\Delta$, makes it possible to find state at next sampling-time $t_{k+1} = (k+1)\Delta$ :

$$x(t_{k+1}) = e^{A(t_{k+1}-t_k)}x(t_k) + \int_{t_k}^{t_{k+1}} e^{A(t_{k+1}-\tau)}Bu(\tau)d\tau \,. \qquad (9)$$

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The ZOH keeps $u(t)$ at a constant value $u(t_k) = u(k\Delta)$, denoted $u(k)$, during time-interval $[t_k, t_{k+1}[$, we then have :

$$u(\tau) = u(k) \text{ pour } \tau \in [t_k, t_{k+1}[\,.$$

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Let us state $\nu = \tau - t_k$, and since $\Delta = t_{k+1} - t_k$, (9) leads to :

$$x(k+1) = e^{A\Delta}x(k) + \int_0^{\Delta} e^{A(\Delta-\nu)}d\nu Bu(k)\,. \tag{10}$$

# Sampled systems : discretizing a continuous-time state-equation

By identifying terms, we obtain :

$$A_{ech} = e^{A\Delta} \tag{11}$$

$$B_{ech} = \int_0^{\Delta} e^{A(\Delta-\nu)} d\nu B \tag{12}$$

$$C_{ech} = C \tag{13}$$

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$$A_{ech} = e^{A\Delta} \quad (11)$$

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$$C_{ech} = C \quad (13)$$

- Matrices $A_{ech}$ and $B_{ech}$ depend on sampling period $\Delta$, and have to be evaluated as soon as the sampling period changes.

# Sampled systems : choice of the sampling period

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- if $\Delta$ is too large, important pieces of information are lacking on the evolution of the system from the computer point of view.
- if $\Delta$ is too small, the computer is excessively "loaded". If the system does evolve only a little bit between two successive samples, the piece of information obtained by a new sample does not bring much and the processor is then requested pointlessly.

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Shannon's

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in which $f_e$ is the sampling frequency and $f_h$ is the highest frequency to be kept into the signal.

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Practical rule applied in control theory

$$5f_h < f_e < 25f_h$$

On one side, it is an "over-sampling" compared to Shannon's theorem. On the other side, this rule gives a bound to avoid to request too often and unnecessarily the processor.

# Sampled systems : choice of the sampling period

Consequently, we select

$$5f_h < f_e < 25f_h \ .$$

| | First-order system | Second-order system |
|---|---|---|
| With | $f_h \approx \frac{1}{2\pi T}$ | $f_h \approx \frac{\omega_n}{2\pi}$ |
| We then get | $0.25T < \Delta < 1.25T$ | $0.25 < \Delta\omega_n < 1.25$ |

# Example : container handling gantry crane

State-space representation and transfer function :

$$\begin{cases} \dot{x}(t) & = & \begin{pmatrix} 0 & g \\ -\frac{1}{L} & 0 \end{pmatrix} x(t) + \begin{pmatrix} 0 \\ \frac{1}{L} \end{pmatrix} u(t) \\ y(t) & = & \begin{pmatrix} 1 & 0 \end{pmatrix} x(t) \end{cases}$$

$$H(s) = C(s\mathbf{I} - A)^{-1}B = \frac{1}{1 + \frac{L}{g}s^2}.$$

It is a second-order system for which the natural frequency is given by $\omega_n = \sqrt{\frac{g}{L}}$. Sampling period $\Delta$ can then be chosen into interval

$$\left] 0.25\sqrt{\frac{L}{g}}, 1.25\sqrt{\frac{L}{g}} \right[$$

# Example : container handling gantry crane

State-space representation of the system sampled at period $\Delta$ :

Using formula $e^{At} = \mathcal{L}^{-1}\left[(s\mathbf{I} - A)^{-1}\right]$, we have[1] :

$$
\begin{aligned}
e^{At} &= \mathcal{L}^{-1}\left[\begin{pmatrix} \frac{s}{s^2+g/L} & \frac{g}{s^2+g/L} \\ -\frac{1/L}{s^2+g/L} & \frac{s}{s^2+g/L} \end{pmatrix}\right] \\
&= \begin{pmatrix} \cos\left(\sqrt{\frac{g}{L}}t\right) & \sqrt{gL}\sin\left(\sqrt{\frac{g}{L}}t\right) \\ -\frac{1}{\sqrt{gL}}\sin\left(\sqrt{\frac{g}{L}}t\right) & \cos\left(\sqrt{\frac{g}{L}}t\right) \end{pmatrix}
\end{aligned}
$$

Hence

$$
A_{ech} = \begin{pmatrix} \cos\left(\sqrt{\frac{g}{L}}\Delta\right) & \sqrt{gL}\sin\left(\sqrt{\frac{g}{L}}\Delta\right) \\ -\frac{1}{\sqrt{gL}}\sin\left(\sqrt{\frac{g}{L}}\Delta\right) & \cos\left(\sqrt{\frac{g}{L}}\Delta\right) \end{pmatrix}
$$

---

1. As a reminder, Laplace transform of :
   - $\sin(\omega t)$ is $\frac{\omega}{s^2+\omega^2}$ ;
   - $\cos(\omega t)$ is $\frac{s}{s^2+\omega^2}$.

# Example : container handling gantry crane

We get [2] $B_{ech} = \int_0^\Delta e^{A(\Delta-\nu)} d\nu B$ :

$$
\begin{aligned}
B_{ech} & \\
= & \int_0^\Delta \begin{pmatrix} \cos\left(\sqrt{\frac{g}{L}}\Delta - \sqrt{\frac{g}{L}}\nu\right) & \sqrt{gL}\sin\left(\sqrt{\frac{g}{L}}\Delta - \sqrt{\frac{g}{L}}\nu\right) \\ -1/\sqrt{gl}\sin\left(\sqrt{\frac{g}{L}}\Delta - \sqrt{\frac{g}{L}}\nu\right) & \cos\left(\sqrt{\frac{g}{L}}\Delta - \sqrt{\frac{g}{L}}\nu\right) \end{pmatrix} d\nu B \\
= & \left[\begin{pmatrix} -\sqrt{\frac{g}{L}}\sin\left(\sqrt{\frac{g}{L}}\Delta - \sqrt{\frac{g}{L}}\nu\right) & -L\cos\left(\sqrt{\frac{g}{L}}\Delta - \sqrt{\frac{g}{L}}\nu\right) \\ \frac{1}{g}\cos\left(\sqrt{\frac{g}{L}}\Delta - \sqrt{\frac{g}{L}}\nu\right) & -\sqrt{\frac{g}{L}}\sin\left(\sqrt{\frac{g}{L}}\Delta - \sqrt{\frac{g}{L}}\nu\right) \end{pmatrix}\right]_0^\Delta B \\
= & \begin{pmatrix} -1 + \cos\left(\sqrt{\frac{g}{L}}\Delta\right) \\ \frac{1}{\sqrt{(gL)}}\sin\left(\sqrt{\frac{g}{L}}\Delta\right) \end{pmatrix}
\end{aligned}
$$

$C_{ech} = C = \begin{pmatrix} 1 & 0 \end{pmatrix}$

---

2. As a reminder, the derivative of :
   - $\frac{1}{\omega}\sin(\omega t + \phi)$ is $\cos(\omega t + \phi)$ ;
   - $-\frac{1}{\omega}\cos(\omega t + \phi)$ is $\sin(\omega t + \phi)$.