Control theory: state-space approach for linear systems

Continuous-time systems

2 janvier 2022

Continuous-time systems

$$\begin{cases} \dot{x}(t) & = & Ax(t) + Bu(t) \\ y(t) & = & Cx(t) \end{cases}$$

Time is assumed to be continuous, that is t takes its values into $\mathbb{R}$.

From a state-space repr. to a TF

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$$\begin{cases} \dot{x}(t) = Ax(t) + Bu(t) \\ y(t) = Cx(t) \\ x(0) = 0 \end{cases} \iff \begin{cases} sX(s) = AX(s) + BU(s) \\ Y(s) = CX(s) \end{cases}$$

$$\iff \begin{cases} (s\mathbf{I} - A)X(s) = BU(s) \\ Y(s) = CX(s) \end{cases}$$

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Finally, we obtain

$$Y(s) = \left[C(p\mathbf{I} - A)^{-1}B\right] U(s) \;\; \text{, or, } \frac{Y(s)}{U(s)} = C(s\mathbf{I} - A)^{-1}B.$$

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$C(s\mathbf{I} - A)^{-1}B$ is the *transfer matrix* of the system (*transfer function* in the single-input single-output case).

Example : container handling gantry crane

State-space representation :

$$\begin{cases} \dot{x}(t) & = & \begin{pmatrix} 0 & g \\ -\frac{1}{L} & 0 \end{pmatrix} x(t) + \begin{pmatrix} 0 \\ \frac{1}{L} \end{pmatrix} u(t) \\ y(t) & = & \begin{pmatrix} 1 & 0 \end{pmatrix} x(t) \end{cases} .$$

We have

$$s\mathbf{I} - A = \begin{pmatrix} s & -g \\ \frac{1}{L} & s \end{pmatrix} , \; |s\mathbf{I} - A| = s^2 + \frac{g}{L} , \; com(s\mathbf{I} - A) = \begin{pmatrix} s & -\frac{1}{L} \\ g & s \end{pmatrix}$$

$$(s\mathbf{I} - A)^{-1} = \frac{1}{|s\mathbf{I} - A|} com(s\mathbf{I} - A)^\top = \frac{1}{s^2 + \frac{g}{L}} \begin{pmatrix} s & g \\ -\frac{1}{L} & s \end{pmatrix} .$$

Then we obtain

$$H(s) = C(s\mathbf{I} - A)^{-1} B = \frac{\frac{g}{L}}{s^2 + \frac{g}{L}} = \frac{1}{1 + \frac{L}{g} s^2} .$$

From a TF to a state-space repr.

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- Consequentlty, from a transfer function there exist several methods to get the distinct but equivalent state-space representations of a system, each of them having a particular form.

From a TF to a state-space repr.

Controllable Canonical Form

$$\begin{cases} \dot{x}(t) = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -a_0 & -a_1 & -a_2 \end{pmatrix} x(t) + \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} u(t) \\ y(t) = \begin{pmatrix} b_0 & b_1 & b_2 \end{pmatrix} x(t) \end{cases} .$$

From a TF to a state-space repr.

Observable Canonical Form

$$
\begin{cases}
\dot{x}(t) = \begin{pmatrix} 0 & 0 & -a_0 \\ 1 & 0 & -a_1 \\ 0 & 1 & -a_2 \end{pmatrix} x(t) + \begin{pmatrix} b_0 \\ b_1 \\ b_2 \end{pmatrix} u(t) \\
\\
y(t) = \begin{pmatrix} 0 & 0 & 1 \end{pmatrix} x(t)
\end{cases}
.
$$

Simulation using a state-space representation

Euler's method

Let dt be a very small number compared to time-constants of the system. dt is the sampling period of the method. The evolution equation can be approximated by

$$\frac{x(t+dt)-x(t)}{dt} \simeq Ax(t) + Bu(t) \quad (1)$$

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or

$$x(t+dt) \simeq x(t) + Ax(t)\cdot dt + Bu(t)\cdot dt \quad (2)$$

Simulation using a state-space representation

Simulation algorithm

```
 x:=x0; t:=0; dt:=0.01;
repeat
     assign its value to u;
     y:=Cx;
     output (display or put into memory) y;
     x:=x+A.x.dt+B.u.dt;
     wait for an interruption from the sampler;
     t=t+dt;
 indefinitely
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Batch mode simulation : line, at which an interruption of the sampler is waited for, is removed from the algorithm (in order to obtain simulation result as fast as possible).

Solution to state-space equations

Theorem

$x(t_0)$ value of the state at initial time-instant t_0, state at t is given by

$$x(t) = e^{A(t-t_0)}x(t_0) + \int_{t_0}^{t} e^{A(t-\tau)}Bu(\tau)d\tau \, , \tag{3}$$

and the output can be written

$$y(t) = Ce^{A(t-t_0)}x(t_0) + \int_{t_0}^{t} Ce^{A(t-\tau)}Bu(\tau)d\tau \, . \tag{4}$$

Solution to state-space equations

Equation (3) gives the state for all $t \geq t_0$ from the initial state $x(t_0)$, and the input $u(t)$ applied along time-interval $[t_0, t]$.

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See appendices for a reminder about matrix exponential.

Stability

Definition

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Assume $u(t) \neq 0$ for $t \in [t_0, t_1]$, state at t_1 is given by :

$$x(t_1) = e^{A(t_1 - t_0)}x(t_0) + \int_{t_0}^{t_1} e^{A(t_1-\tau)}Bu(\tau)d\tau \, .$$

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If $u(t) = 0$ for $t > t_1$, state for $t \geq t_1$ can be written :

$$\begin{aligned} x(t) &= e^{A(t - t_1)} x(t_1) + \int_{t_1}^{t} e^{A(t - \tau)} B u(\tau) d\tau \\ &= e^{A(t - t_1)} x(t_1) \, , \end{aligned}$$

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which tends to zero while $t \to \infty$ if $\lim_{t\to\infty} e^{A(t-t_1)} = \mathbf{0}$.

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Theorem

A linear system is said to be stable if, and only if, all the eigenvalues of its matrix evolution have a strictly negative real part.

Stability

Definition

The *characteristic polynomial* $P(s)$ of a linear system is defined by

$$P(s) = |s\mathbf{I} - A| \tag{5}$$

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Single-input single-output case : $P(s)$ is the denominator of the transfer function of the system, and its roots correspond to the *poles*.

Corollary (Stability criterion)

A linear system is said to be stable if, and only if, all the roots of its characteristic polynomial have a strictly negative real part.

Example : container handling gantry crane

State-space representation :

$$\left\{ \begin{array}{lll} \dot{x}(t) & = & \begin{pmatrix} 0 & g \\ -\frac{1}{L} & 0 \end{pmatrix} x(t) + \begin{pmatrix} 0 \\ \frac{1}{L} \end{pmatrix} u(t) \\ y(t) & = & \begin{pmatrix} 1 & 0 \end{pmatrix} x(t) \end{array} \right. .$$

From this state-space representation, stability of the system can be decided by evaluating the eignevalues of A which are also the roots of its characteristic polynomial :

$$P(s) = |s\mathbf{I} - A| = \begin{vmatrix} s & -g \\ \frac{1}{L} & s \end{vmatrix} = s^2 + \frac{g}{L}.$$

These roots are pure imaginary numbers $\pm i\sqrt{\frac{g}{L}}$. The system is consequently unstable.

Controllability

Definition (Controllability)

A linear system is said to be controllable if for any pair of state-vectors (x_0, x_1), one can find a time-instant t_1 and an input $u(t)$, $t \in [t_0, t_1]$, such that the system, initialized with x_0 at time t_0, can reach the state x_1 at t_1.

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Theorem (Controllability criterion)

A linear system is said to be controllable if, and only if,

$$\operatorname{rank} \overbrace{\left(B|AB|A^2B|\ldots|A^{n-1}B\right)}^{\Gamma_{com}} = n\,, \tag{6}$$

where n is the dimension of matrix evolution A.

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We have

$$\Gamma_{com} = \begin{pmatrix} B & AB \end{pmatrix} = \begin{pmatrix} 0 & \frac{g}{L} \\ \frac{1}{L} & 0 \end{pmatrix}$$

The determinant of Γ_{com} is not equal to zero, this implies that the rank of Γ_{com} is then equal to 2 (the dimension of A), and we can conclude that the system is controllable. This means that, from any initial value of the state (state variables are the mass speed and angle θ), it is possible to give to the state any desired value.

Observability

Definition (Observability)

A linear system is said to be observable if the knowledge of $y(t)$ and $u(t)$ for $t \in \mathbb{R}$ is sufficient to determine state $x(t)$, for all t.

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A linear system is said to be observable if, and only if,

$$\mathrm{rang} \begin{pmatrix} C \\ CA \\ \vdots \\ CA^{n-1} \end{pmatrix} = n. \tag{7}$$

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We have

$$\Gamma_{obs} = \begin{pmatrix} C \\ CA \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & g \end{pmatrix},$$

$|\Gamma_{obs}| = g \neq 0$ and then $\mathrm{rang}(\Gamma_{obs}) = 2 = \dim(A)$. The system is therefore observable. This means that the knowledge of the input and the output is sufficient to be able to evaluate the state variables.