# Control theory: state-space approach for linear systems

## Preliminary review

2 janvier 2022

# Dynamical model and system

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- *linear*, if the *superposition principle* applies :

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- *stationnary* if the relations between the input and the output remain the same as time ellapses (no variation of system- parameters).
- *causal* if system output at any time instant $t_0$, $y(t_0)$, does not depend on future values of input $u(t)$, $t > t_0$ (all the physical systems are causal).

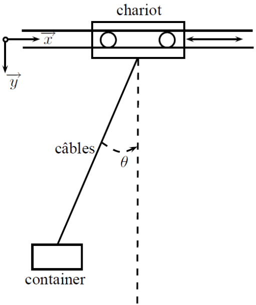
# Example : container handling gantry crane

FIGURE – container handling gantry crane

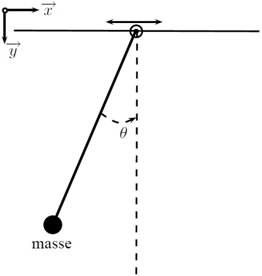
We focus on horizontal shifting of set trolley-cables-container (lifting is stopped).
We aim at **regulating the container speed while avoiding the container to swing (dangling)**.

# Example : container handling gantry crane

- Set trolley-cables-container modeled as a pendulum with a mobile pivot
- Cables $\equiv$ indeformable solid, in pivot-connection with the trolley
- Container $\equiv$ punctual mass placed in the center of gravity
- Frictions are neglected

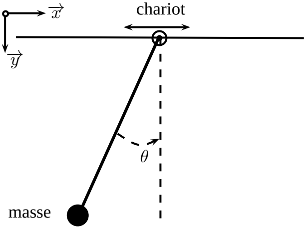



Schématisation de l'ensemble chariot-câbles-container




Abstraction sous la forme d'un pendule

# Example : container handling gantry crane




| | |
|---|---|
| $m$ | mass of the container |
| $L$ | cables length ($\equiv$ indeformable solid) |
| $g$ | gravity acceleration |

| | |
|---|---|
| $p_m, v_m, a_m$ | position, speed and acceleration of the mass projected onto axis $\overrightarrow{x}$ |
| $p_c, v_c, a_c$ | position, speed and acceleration of the trolley projected onto axis $\overrightarrow{x}$ |
| $\theta$ | angle between cables and the vertical axis |

# Example : container handling gantry crane

The goal is to regulate the mass-speed while avoiding it to swing. We consider that :

- the **input** of the system is the horizontal speed of the trolley $v_c$ (at any time instant $t$, we assume to be able to set the value of $v_c(t)$) ;
- the **output** of the system is the horizontal speed $v_m$ of the mass (container).

We neglect **disturbances (noises)** influencing the system : we could take into account the wind (uncontrolled exogeneous quantity influencing the container-dangling).

This system is **deterministic** : an input $v_c$ leads to only one possible output $v_m$.

This system is **causal** : the output value at a time instant $t_0$, $v_m(t_0)$,does not depend on any future value of the input $v_c(t)$ for $t > t_0$.

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**Modelling** of this system : to obtain the differential equations (parametric model) describing the evolution of variables :

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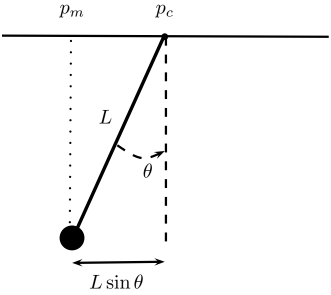
We have $p_c = p_m + L \sin\theta$, hence the trolley speed is related to the mass-speed by :

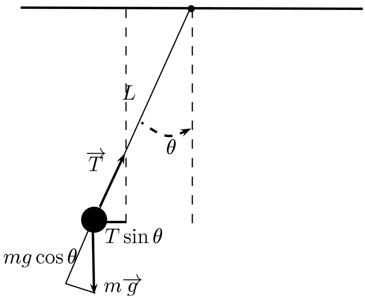
$$v_c = v_m + L\frac{d\sin\theta}{dt}. \quad (1)$$



# Example : container handling gantry crane

Fundamental principle of the dynamics (Newton's second law) applied to the mass and projected onto $\overrightarrow{x}$ gives : $ma_m = T\sin\theta$.
Cables considered as indeformables $\Rightarrow T = mg\cos\theta$.
We deduce that

$$ma_m = mg\sin\theta\cos\theta \tag{2}$$



# Example : container handling gantry crane

System modeled by the equations

$$v_c = v_m + L\frac{d\sin\theta}{dt}$$

$$ma_m = mg\sin\theta\cos\theta$$

- **stationnary** since the parameters in these differential equations (that is $m$, $g$ and $L$) are assumed to be constant as time ellapses.
- **not linear**.

# Example : container handling gantry crane

System modeled by

$$v_c = v_m + L\frac{d\sin\theta}{dt} \qquad ma_m = mg\sin\theta\cos\theta$$

To have a linear model, we assume approximations on the behavior of the system (less precise model but easier to use since linear) : we consider that $\theta \approx 0$ and so $\cos\theta \approx 1$, $\sin\theta \approx \theta$. We deduce the following linear model :

$$\frac{d\theta}{dt} = -\frac{1}{L}v_m + \frac{1}{L}v_c \tag{3}$$

$$\frac{dv_m}{dt}(= a_m) = g\theta \tag{4}$$

# Reminders about linear systems

| | CONTINUOUS TIME |
|---|---|
| Convolution | $y(t) = (h * u)(t) = \int_0^t h(\tau)u(t-\tau)d\tau$ |
| Transfer F. | $H(s) = \frac{b_m s^m + b_{m-1}s^{m-1} + \ldots + b_1 s + b_0}{s^n + a_{n-1}s^{n-1} + \ldots + a_1 s + a_0}$ <br> $\mathcal{L}[h(t)] = H(s)$ <br> $Y(s) = H(s)U(s)$ |
| Diff. Equ. | $y^{(n)} + a_{n-1}y^{(n-1)} + \ldots + a_1\dot{y} + a_0 y$ <br> $= b_m u^{(m)} + b_{m-1}u^{(m-1)} + \ldots + b_1\dot{u} + b_0 u$ <br> C.I. $y(0^+), \dot{y}(0^+), \ldots, y^{(n-1)}(0^+)$ |

# Example : container handling gantry crane

Laplace transform of equations

$$\frac{d\theta}{dt} = -\frac{1}{L}v_m + \frac{1}{L}v_c, \qquad \frac{dv_m}{dt}(= a_m) = g\theta,$$

is

$$s\Theta(s) = -\frac{1}{L}V_m(s) + \frac{1}{L}V_c(s), \qquad sV_m(s) = g\Theta(s),$$

assuming $\dot{\theta}(0) = 0$, $v_m(0) = 0$, $v_c(0) = 0$ for initial conditions.

A transfer function can then be deduced.

# Example : container handling gantry crane

We have obtained :

$$\begin{cases} s\Theta(s) & = & -\frac{1}{L}V_m(s) + \frac{1}{L}V_c(s) \\ sV_m(s) & = & g\Theta(s) \Longleftrightarrow \Theta(s) = \frac{s}{g}V_m(s) \end{cases}$$

This leads to

$$\frac{s^2}{g}V_m(s) = -\frac{1}{L}V_m(s) + \frac{1}{L}V_c(s)$$

$$\Longleftrightarrow \left(\frac{s^2}{g} + \frac{1}{L}\right)V_m(s) = \frac{1}{L}V_c(s)$$

Hence

$$\frac{V_m(s)}{V_c(s)} = \frac{1}{1 + \frac{L}{g}s^2} = \frac{Y(s)}{U(s)} = H(s).$$

# Example : container handling gantry crane

The obtained transfer function

$$H(s) = \frac{Y(s)}{U(s)} = \frac{V_m(s)}{V_c(s)} = \frac{1}{1 + \frac{L}{g}s^2}$$

is a second-order ones whose standard form is

$$H(s) = K\frac{1}{1 + 2\xi\frac{s}{\omega_n} + (\frac{s}{\omega_n})^2}$$

By identification, we can deduce $\omega_n = \sqrt{\frac{g}{L}}$ and $\xi = 0$.

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## Theorem (Stability criterion regarding the transfer function)

*Let a system be described by its transfer function $H(s)$. It is stable if, and only if, the poles (i.e. the roots of the denominator of $H(s)$) have all a strictly negative real part.*

# Example : container handling gantry crane

We can guess that the system corresponding to the set trolley-cables-container is not stable. For example, if the input is first set to a non-zero given value (the trolley-speed is set to a non-zero value), and then set to zero (the trolley is stopped) : in that case we can expect that the container will swing and these oscillations will continue indefinitely (frictions are neglected so that the range of the oscillations will not decrease) which means that the output-response will not tend to zero asymptotically.

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$$H(s) = \frac{1}{1 + \frac{L}{g}s^2}$$

The roots of $1 + \frac{L}{g}s^2$ (denominator of $H(s)$, the transfer function) are $\pm i\sqrt{\frac{g}{L}}$. Their real-part is not strictly negative. This demonstrates that the system is unstable.

# Precision

A controlled-system with output $y(t)$ is all the more precise that the difference between the desired output $y_d(t)$ and the actual output $y(t)$ is low. The *precision* can be quantified by :

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Sometimes, the *stationary error* is considered. The stationary error of order $n$, denoted $\varepsilon_n$, is the steady-state error for an input of the form $U(s) = \frac{1}{s^n}$.