SYSTEM IDENTIFICATION

system characterization and parameter estimation techniques

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Part I

Preliminaries

Outline of Part I

Motivations

Building a model. What for?

Model : mathematical description of the dynamic behavior of a real process (*system*).

Several possible goals

- To analyze: to find system's properties, to deduce some expectable performances, to improve knowledge of the *phenomena* involved . . . ;
- To simulate: to predict evolution according to given *scenarios*, to teach (flight simulator, nuclear power plant simulator, . . .) ;
- To control: to design controllers applied to systems such that no human inputs are needed for correction (e.g. cruise control for regulating a car's speed).

The goal pursued conditions the model: accuracy, complexity, . . .

Modeling *vs* Identification

Two distinct approaches in order to build a model.

Modeling

One can build a so-called *white-box model* by describing precisely phenomena occurring in the system. Depending on the type of the system, we then use laws of Physics, of Chemistry, of Biology. . . . For example, a model for a physical process can be established from the Newton equations.

Identification

Starts from measurements of the behavior of the system and the external influences (inputs to the system) and tries to determine a mathematical relation between them without going into the details of what is actually happening inside the system (*black-box model*).

Modeling *vs* Identification (cont'd)

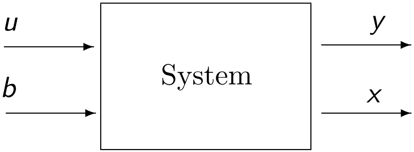
Remarks

- Sometimes, these approaches are mixed to obtain a so-called *grey-box model*.
- In a white-box model, parameters and variables have an interpretation, that is a "physical meaning".
- On the contrary, the variables and the parameters of a black-box model do not necessarily correspond to quantities which can be discerned.

Definitions

System

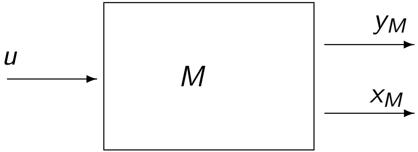
Part of the universe that is (arbitrarily) studied as a whole set.



- We watch (measure) the *outputs* y.
- We are interested by the *state variables* x. The outputs are state variables which are measured.
- We influence the *inputs* u.
- The system is subjected to *disturbances* or *noises* b.

Model

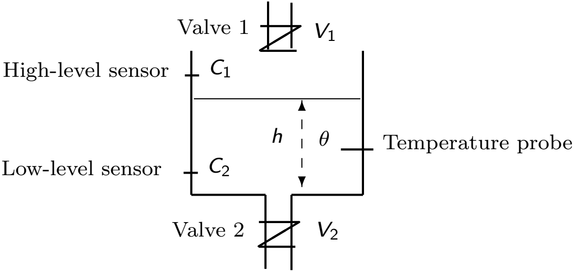
Mathematical description of a system.



From u (the same input as the system's ones), we can deduce y_M and x_M that we expect to be close to y and x.

A classification for systems (and for models)

Example



We pay attention to the following quantities (variables) :

h : level of liquid,

θ : temperature of the liquid,

V_i : boolean state of the valve V_i,

C_i : boolean state of the sensor C_i.

Discrete variable, continuous variable

Def1: A variable is said to be *discrete* if it takes values in a *countable (denumerable)* set (possibly infinite).

Ex: $V_i \in \{0, 1\}$ and $C_i \in \{0, 1\}$.

Def2: A variable is said to be *continuous* if it takes values in an *uncountable (nondenumerable)* set and if its behavior does not have any discontinuity.

Ex: We may consider that $h \in \mathbb{R}$ and $\theta \in \mathbb{R}$.

Def3: A variable taking its values in an uncountable set and whose behavior has discontinuities, is said to be *hybrid*.
A hybrid variable is piecewise continuous: it can be described by means of a continuous variable on each piece and an additional discrete variable can be used to take into account a discontinuity.

Ex: Speed of a pool ball hitting a rail.

Continuous system, discrete event system, hybrid system

Def4:

a) A system is said to be a *Discrete Event (Dynamic) System* (DE(D)S) if all its state variables are discrete.
b) A system is said to be a *Continuous System* (CS) if all its state variables are continuous.
c) A system is said to be an *Hybrid (Dynamic) System* (H(D)S) if it has at least one discrete variable and at least one continuous variable.

Ex:

a) If we consider as state variables $\begin{pmatrix} V_1 & V_2 & C_1 & C_2 \end{pmatrix}^\top$, then it is a DES.
b) If we consider as state variables $\begin{pmatrix} h & \theta \end{pmatrix}^\top$, then it is a CS.
c) If we consider as state variables $\begin{pmatrix} h & V_1 & V_2 \end{pmatrix}^\top$, then it is a HS.

CS, DES, HS (Remarks)

- The goal for modeling-identification imposes the type of model. If we only want to establish the logical control of valves in order to keep the level of liquid between the high and low sensors, then we can use a model with $\begin{pmatrix} V_1 & V_2 & C_1 & C_2 \end{pmatrix}^\top$ as state vector (DES). If we want to describe the evolution of $h(t)$ when $V_1 = 1$ and $V_2 = 0$, we then use a model with h as state with additional parameters such as the flow through V_1 and the surface area of the tank.

CS, DES, HS (Remarks)

- A sampled continuous system is still a continuous system (discrete-time).
- In a DES, the state transitions are instantaneous and correspond to *events*. This explains the used term *Discrete* *Event* *System*.

Scope of the course

Scope of the course

We will focus our attention on

- **identification** (the course "Modeling and Simulation" (3A) is devoted to modeling)
- **of continuous systems** (a 4A-course deals with Discrete Event Systems).

Part II

Introduction to the identification of continuous systems

Outline of Part II

Reminders on models for continuous systems

Reminders on continuous time models

Convolution

$$y(t) = (h * u)(t) = \int_0^t h(\tau)u(t-\tau)d\tau$$

Transfer Function

$$H(s) = \frac{b_n s^n + b_{n-1}s^{n-1} + \ldots + b_1 s + b_0}{s^n + a_{n-1}s^{n-1} + \ldots + a_1 s + a_0} = \mathcal{L}[h(t)]$$

$$Y(s) = H(s)U(s)$$

- *Order*: degree of the denominator
- *Degree*: degree of the numerator (often b_n=0).

Reminders on continuous time models (ctd)

Differential equation

$$y^{(n)} + a_{n-1}y^{(n-1)} + \ldots + a_1\dot{y} + a_0 y$$
$$= b_n u^{(n)} + b_{n-1}u^{(n-1)} + \ldots + b_1\dot{u} + b_0 u$$

With initial conditions $y(0^+), \dot{y}(0^+), \ldots, y^{(n-1)}(0^+)$

- $\exists$ adequate tranformations from one model to the other
- n is a minimal value for the size of x

Reminders on discrete time models

Convolution

$$y(k) = (h * u)(k) = \sum_{i=0}^{k} h(i)u(k-i)$$

Transfer function

$$H(z) = \frac{b_n z^n + b_{n-1} z^{n-1} + \ldots + b_1 z + b_0}{z^n + a_{n-1} z^{n-1} + \ldots + a_1 z + a_0} = \mathcal{Z}[h(k)]$$

$$Y(z) = H(z)U(z)$$

Reminders on discrete time models (ctd)

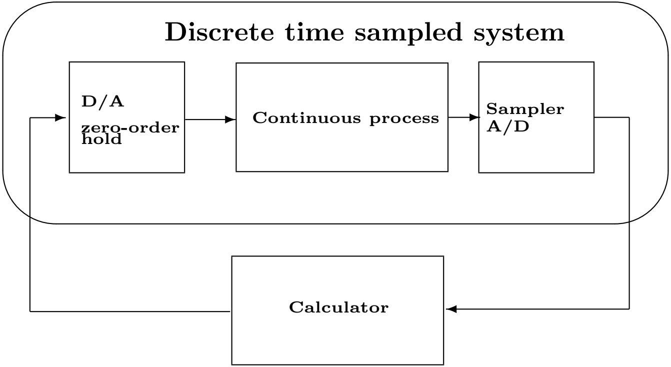
Recursive equation

$$y(k+n) + a_{n-1}y(k+n-1) + \ldots + a_0 y(k) = b_n u(k+n) + b_{n-1}u(k+n-1) + \ldots + b_0 u(k)$$

I.C. $y(0), y(1), \ldots, y(n-1)$

Reminders on discrete time models (ctd)

Continuous system studied as a discrete-time system by means of a calculator



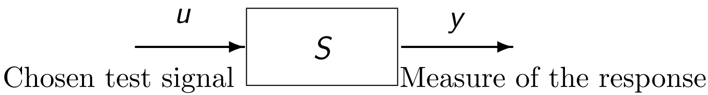
- $\exists$ adequate tranformations from continuous model to discrete time model

Identification issue

Assumptions

linear and mono-variable models

Issue



$\mathcal{S}$ linear and single-input single-output

How to establish a parametric model of S?

Identification methodology

Two steps

Two main steps in the identification procedure:

- *choose a structure for the model*: **characterization** (Part III);
- *estimate the parameters numeric values in the model*: **estimation** (Part IV).

Part III

Identification: characterization of continuous systems

Characterization

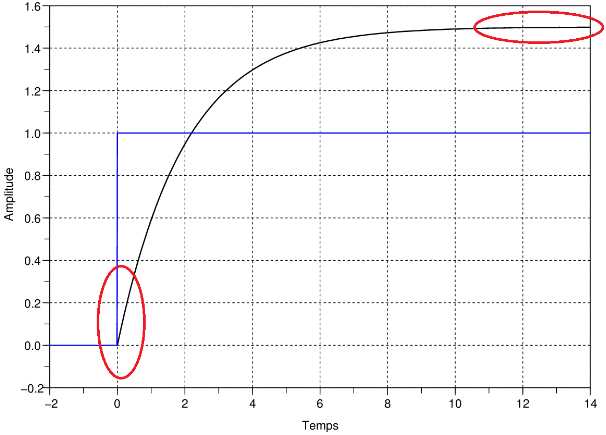
Characterization

The object here is to choose the structure for the model, that is, typically to determine the value of n (*order* of the model).

Outline of Part III

Characterization by means of a step response

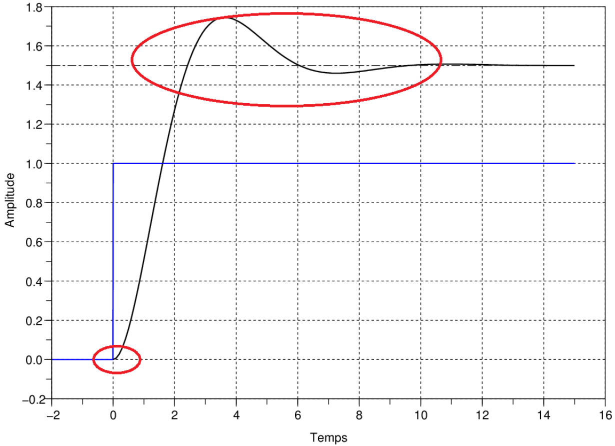
First-order system



Properties

non-zero derivative, no overshoot

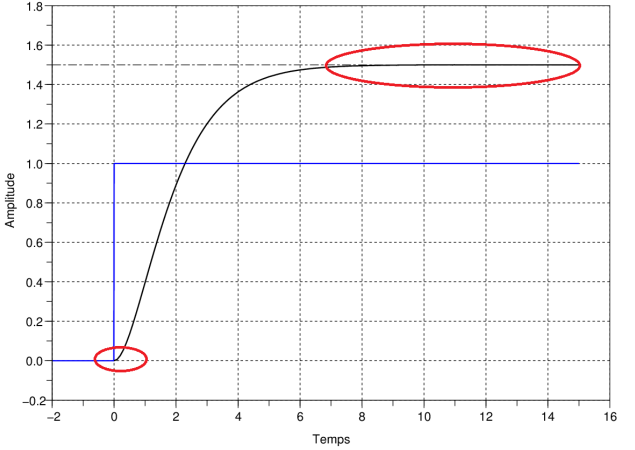
Second-order system



Properties

zero derivative, possible overshoot(s)

Second-order system



Properties

zero derivative, possible overshoot(s) but not necessarily !

n-order system (with $n > 2$)

Difficult characterization by means of a step response

- Various possible shapes
- Superimposition of responses of first and second order subsystems

Possible approaches

- Approximate the system as a first-order or second-order system
- Use a more sophisticated method

Systems with ideal delay

There exist systems for which there is a time-delay between the input applied and the output response of the system.

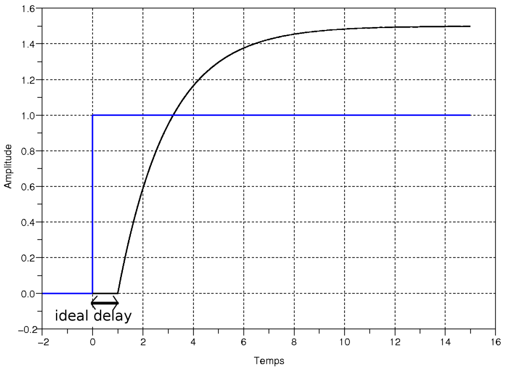


Figure: Step response of a first order system with an ideal time-delay

Systems with ideal delay

Denoting τ the value of the time-delay, and r the number of discrete-time step(s) corresponding to τ, we have for n-order systems:

Continuous-time	Discrete-time
$H(s) = \frac{b_n s^n + \ldots + b_0}{s^n + \ldots + a_0} \mathbf{e}^{-\mathbf{s}\tau}$	$H(z) = K \frac{b_n z^n + \ldots + b_0}{z^n + \ldots + a_0} \mathbf{z}^{-\mathbf{r}}$

- for a discrete time sampled system $r = \lfloor \tau / \Delta \rfloor$

Systems with ideal delay

Let us point out that

$$\frac{b_n z^n + b_{n-1} z^{n-1} \ldots + b_0}{z^n + a_{n-1} z^{n-1} + \ldots + a_0} \mathbf{z^{-r}}$$

$$= \frac{\beta_{n+r} z^{n+r} + \beta_{n+r-1} z^{n+r-1} + \ldots + \beta_0}{z^{n+r} + \alpha_{n+r-1} z^{n-1+r} + \ldots + \alpha_1 z + \alpha_0}$$

with

- $\beta_{n+r} = \ldots \beta_{n+1} = 0$ and $\beta_n = b_n, \ldots, \beta_0 = b_0$,
- $\alpha_{n+r-1} = a_{n-1}, \ldots, \alpha_r = a_0$ and $\alpha_{r-1} = \ldots = \alpha_0 = 0$.

$\Rightarrow$ a n-order system with r ideal delays can be studied by means of a $n + r$-order model

Characterization using a random signal

Motivations

Order determination using a step response

- May be inaccurate: based on observations of the curve-shape (with possible noises on its measure)
- Limited to n-order systems with $n \leq 2$

Order determination by means of quotient of instrumental determinants (QID) test

- Use a response to any (random) input
- Determine the order of a system in a "systematic" manner
- Make it possible to evaluate order n even with $n > 2$

Order determination by means of QID test

Consider N measures: $u(i)$ and $y(i)$, $i = 1, \dots N$.

Information matrix Q_i at step i is built according to:

$$Q_i = \frac{1}{N} \sum_{k=i}^{N-i} \begin{pmatrix} u(k) \\ u(k+1) \\ u(k-1) \\ u(k+2) \\ \vdots \\ u(k-i+1) \\ u(k+i) \end{pmatrix} \begin{pmatrix} y(k+1) & u(k+1) & \dots & y(k+i) & u(k+i) \end{pmatrix}.$$

Q_i is a square $2i \times 2i$ matrix.

Q_i **has full rank for** $i \leq n$**, is almost rank deficient otherwise.**

Order determination by means of QID test

The **quotient of instrumental determinants** (QID) is defined by

$$QID(i) = \frac{|Q_i|}{|Q_{i+1}|}$$

Procedure

For $i = 1, \dots, M$ (with n supposed to be less than M)

- Build information matrices Q_i and Q_{i+1},
- Evaluate quotient of instrumental determinants $QID(i)$.

The order of the system is the value of i for which the absolute value of quotient $QID(i)$ increases suddenly for the first time.

Part IV

Identification: parameter estimation techniques

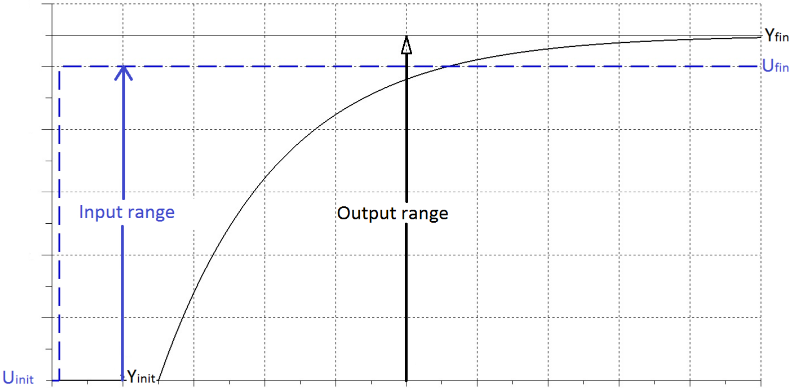
Outline of Part IV

Estimation by means of a step response

Static gain

Ratio of the output and the input under steady state condition:

$$K = \frac{\text{Output range}}{\text{Input range}} = \frac{y_{fin} - y_{init}}{u_{fin} - u_{init}}$$



First-order system

Continuous time

$$H(s) = K\frac{1}{1+Ts}$$

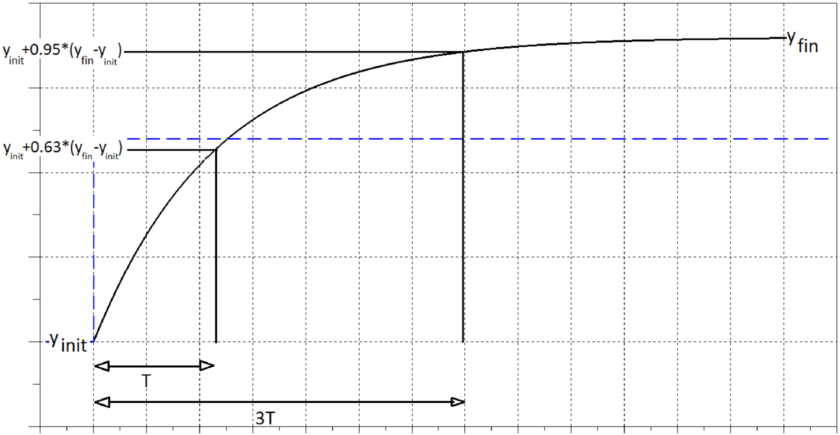
Discrete time (sampled system)

Choice of the sampling period: $0.25\,T < \Delta < 1.25\,T$

$$\begin{aligned} H(z) &= \tfrac{z-1}{z}\mathcal{Z}\left[\tfrac{H(s)}{s}\right] \\ &= K\tfrac{1-z_0}{z-z_0} \text{ with } z_0 = e^{-\frac{\Delta}{T}} \end{aligned}$$

First-order system

Time-constant T can be determined according to:



Second-order system

Continuous time

$$H(s) = K \frac{1}{1 + 2\xi \frac{s}{\omega_n} + (\frac{s}{\omega_n})^2}$$

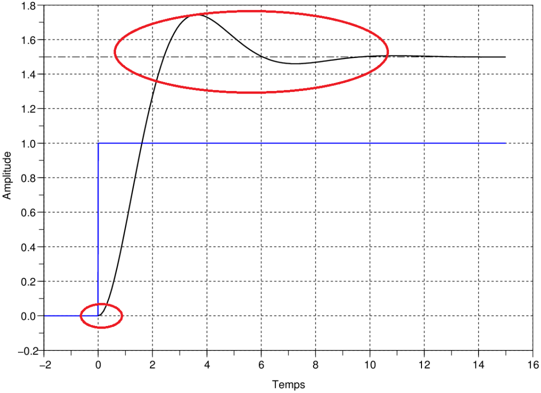
ξ	damping ratio
ω_n	natural frequency

Discrete time

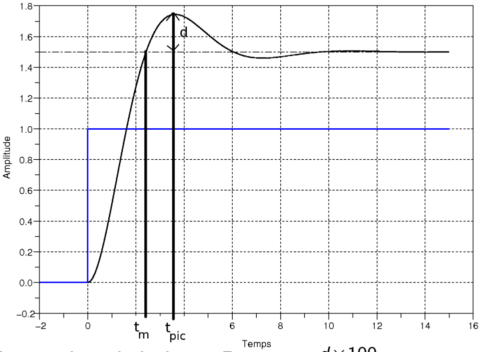
$$H(z) = \frac{b_1 z + b_0}{z^2 + a_1 z + a_0}$$

Second-order system (ctd)

Under-damped second-order system ($0 < \xi < 1 \Rightarrow$ overshoot(s))



Second-order system (ctd)



1. Measure overshoot d and deduce $D_{\%} = \frac{d \times 100}{\text{Output range}}$
2. Deduce ξ from expression : $D_{\%} = 100 e^{-\pi\xi/\sqrt{1-\xi^2}}$
3. Deduce ω_n from expression of *rise time* t_m or *peak time* t_{peak}:
   $t_m = \frac{1}{\omega_n\sqrt{1-\xi^2}}(\pi - \arccos\xi)$ $\qquad t_{peak} = \frac{\pi}{\omega_n\sqrt{1-\xi^2}}$

Second-order system (ctd)

Under-damped second-order system ($0 < \xi < 1 \Rightarrow$ overshoot(s))

Discrete time sampled system

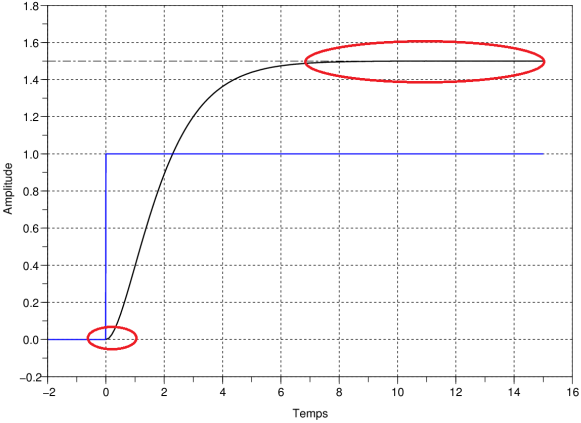
$$H(z) = \frac{b_1 z + b_0}{z^2 + a_1 z + a_0}$$

- Choice of the sampling period: $0.25 < \omega_n \Delta < 1.25$.
- Denoting $\alpha = e^{-\xi\omega_n\Delta}$ and $\omega_p = \omega_n\sqrt{1-\xi^2}$, we have

$$a_0 = \alpha^2$$
$$a_1 = -2\alpha\cos(\omega_p\Delta)$$
$$b_0 = \alpha^2 + \alpha\left[\xi\frac{\omega_n}{\omega_p}\sin(\omega_p\Delta) - \cos(\omega_p\Delta)\right]$$
$$b_1 = 1 - \alpha\left[\xi\frac{\omega_n}{\omega_p}\sin(\omega_p\Delta) + \cos(\omega_p\Delta)\right]$$

Second-order system (ctd)

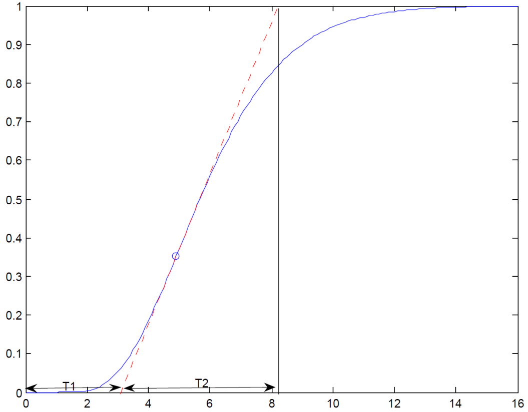
Overdamped second-order system ($\xi \geq 1 \Rightarrow$ no overshoot)



Second-order system (ctd)

Overdamped second-order system ($\xi \geq 1$)

$$H(s) = \frac{K}{(1+T_1 s)(1+T_2 s)}, \text{ with } \xi = \frac{1}{2}\frac{T_1+T_2}{\sqrt{T_1+T_2}} \text{ and } \omega_n = \frac{1}{\sqrt{T_1 T_2}}$$



Second-order system (ctd)

Overdamped second-order system ($\xi \geq 1$)

Discrete time sampled system

$$H(z) = \frac{b_1 z + b_0}{z^2 + a_1 z + a_0}$$

- Choice of the sampling period: $0.25 < \omega_n \Delta < 1.25$.
- Denoting $z_1 = e^{-\Delta/T_1}$ and $z_2 = e^{-\Delta/T_2}$, we have
  $$a_0 = z_1 z_2$$
  $$a_1 = -(z_1 + z_2)$$
  $$b_0 = K\left(z_1 z_2 - \frac{T_1 z_2 - T_2 z_1}{T_1 - T_2}\right)$$
  $$b_1 = K\left(\frac{1 - T_1 z_1 - T_2 z_2}{T_1 - T_2}\right)$$

n-order system (with $n > 2$)

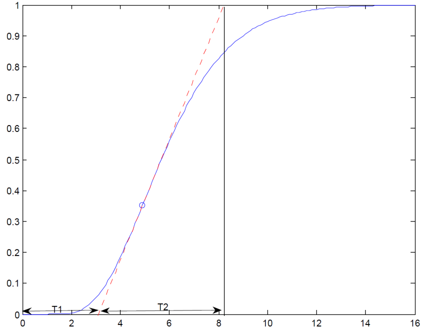
Difficult estimation by means of a step response

- Various possible shapes
- Superimposition of responses of first and second order subsystems

Possible approaches

- Approximate the system as a first-order or second-order system
- Use a more sophisticated method (continuation of the course)
- If no overshoot, *Strejc*'s method can be used

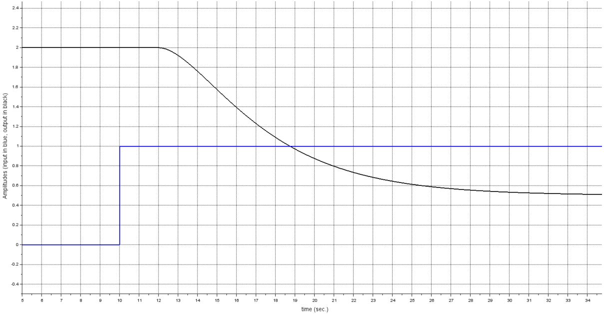
n-order system (*Strejc*'s method)



- deduce n from ratio T_1/T_2 and T from ratio T_1/T or T_2/T

n	$\frac{T_1}{T}$	$\frac{T_2}{T}$	$\frac{T_1}{T_2}$
3	0,8	3,7	0,22
4	1,42	4,46	0,32
⋮	⋮	⋮	⋮

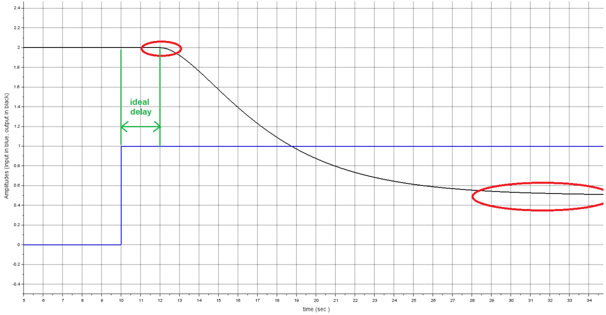
Identification from a step response: an example



The data used may be less "academic" :

- the output of the system can be non-zero before the input step is applied,
- the step can be applied at a time different from $t = 0$,
- the gain can be negative,
- there can be noises

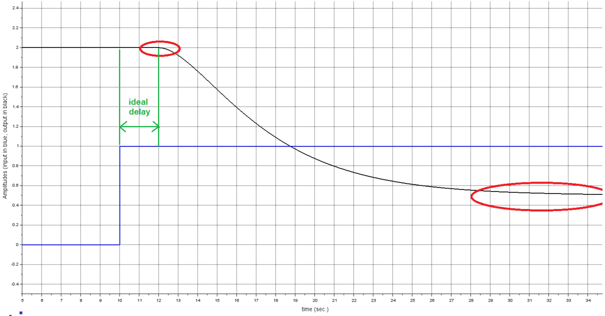
Identification from a step response: an example (Ctd)



Characterization

- the derivative at the beginning of the response seems to be zero and there is no overshoot : aperiodic 2nd order system
- there is a delay between the application of the step and the start of the response

Identification from a step response: an example (Ctd)



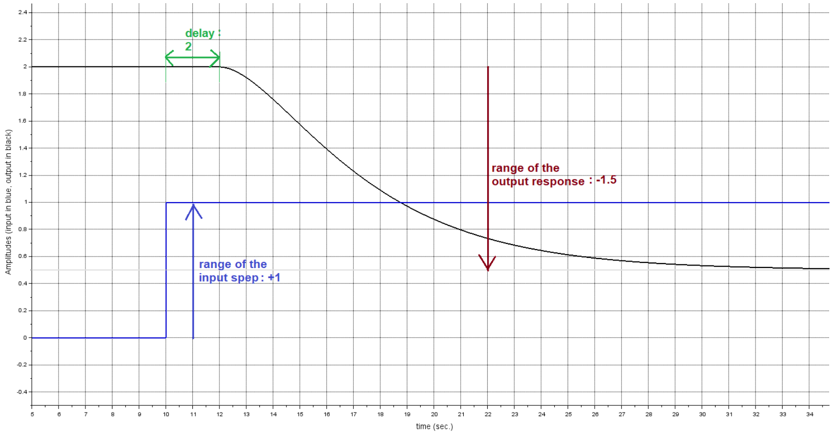
Characterization

- Aperiodic 2nd order system
- ideal delay

An adapted continous-time transfer function is :

$$H(s) = K \frac{1}{1 + 2\xi \frac{s}{\omega_n} + (\frac{s}{\omega_n})^2} e^{-s\tau} = \frac{K}{(1 + T_1 s)(1 + T_2 s)} e^{-s\tau}$$

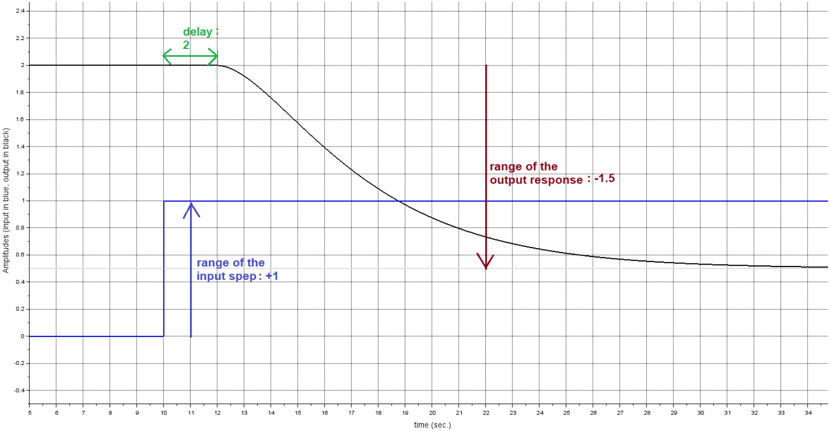
Identification from a step response: an example (Ctd)



Identification

- The delay τ can be easily read: $\tau = 2$ sec.
- The gain K is here negative : $K = \frac{Y_{fin} - Y_{init}}{U_{fin} - U_{init}} = \frac{0.5 - 2}{1 - 0} = -1.5$

Identification from a step response: an example (Ctd)



Identification

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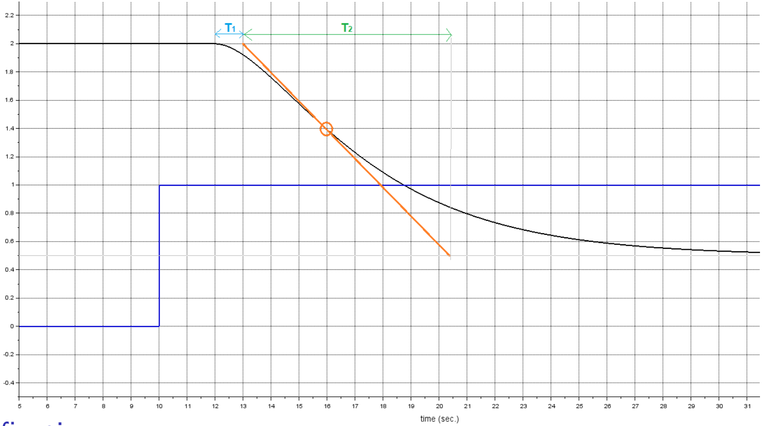
Identification from a step response: an example (Ctd)



Identification

- The tangent is drawn at the inflection point of the output response.
- The intersections with the horizontal lines of initial and final value allow us to identify T_1 and T_2. Be careful to exclude the initial delays and the ideal delay in the measures (the same goes for rise time, peak time,...)

Estimation using a random signal

Off-line estimation procedure

0) Characterization: choice of order n

$$y_M(i+n) = -a_{n-1}y_M(i+n-1) - \ldots - a_0 y_M(i) + b_n u(i+n) + \ldots + b_0 u(i)$$

Off-line estimation procedure

0) Characterization: choice of order n

$$y_M(i+n) = -a_{n-1}y_M(i+n-1) - \ldots - a_0 y_M(i) + b_n u(i+n) + \ldots + b_0 u(i)$$

1) Experiment

N measures :

- $u(i)$, $i = 1, \ldots, N$ is the chosen signal applied in input (typically a PRBS for Pseudo Random Binary Sequence)
- $y(i)$, $i = 1, \ldots, N$ is the measure of the system response

Global procedure estimation

0) Characterization: choice of order n

$$y_M(i+n) = -a_{n-1}y_M(i+n-1) - \ldots - a_0 y_M(i) + b_n u(i+n) + \ldots + b_0 u(i)$$

1) Experiment : N measures of u and y

2) Off-line estimation

Find the parameters vector

$$\theta^\top = \begin{pmatrix} a_0 & \ldots & a_{n-1} & b_0 & \ldots & b_n \end{pmatrix}$$

such that model output $y_M(i)$ is close to $y(i)$ for $i = 1, \ldots, N$, if $u(i)$ is applied as input of the model, that is by minimizing **output error** $\varepsilon = y(i) - y_M(i)$.

Ordinary least square method

Notations

- $\hat{y}(i)$ output estimated by

$$\hat{y}(i+n) = -a_{n-1}y(i+n-1) - \ldots - a_0 y(i) + b_n u(i+n) + \ldots + b_0 u(i),$$

$$\Longleftrightarrow$$

$$\hat{y}(i) = -a_0 y(i-n) - \ldots - a_{n-1}y(i-1) + b_0 u(i-n) + \ldots + b_n u(i),$$

- $\epsilon(i) = y(i) - \hat{y}(i)$ the equation error.

Ordinary least square method

Notations

- $\hat{y}(i)$ output estimated by

$$\hat{y}(i+n) = -a_{n-1}y(i+n-1) - \ldots - a_0 y(i) + b_n u(i+n) + \ldots + b_0 u(i),$$

$$\Longleftrightarrow$$

$$\hat{y}(i) = -a_0 y(i-n) - \ldots - a_{n-1} y(i-1) + b_0 u(i-n) + \ldots + b_n u(i),$$

- $\epsilon(i) = y(i) - \hat{y}(i)$ the equation error.

Do not confuse

- for output error $\varepsilon(i)$, output $y_M(i)$ is evaluated by means of previous values of the model output $y_M(i-1), \ldots, y_M(i-n)$,
- for equation error $\epsilon(i)$, estimated output $\hat{y}(i)$ is evaluated by means of previous measures $y(i-1), \ldots, y(i-n)$. Actually, it is this error that we shall attempt to minimize.

Ordinary least square method

We have $\epsilon(i) = y(i) - \hat{y}(i)$ with

$$\hat{y}(i) = -a_0 y(i-n) - \ldots - a_{n-1} y(i-1) + b_0 u(i-n) + \ldots + b_m u(i).$$

Hence,

$$\begin{aligned} y(i) &= \hat{y}(i) + \epsilon(i) \\ &= -a_0 y(i-n) - \ldots - a_{n-1} y(i-1) + b_0 u(i-n) + \ldots + b_m u(i) + \epsilon(i). \end{aligned}$$

This last equation rewritten for $i = n+1, n+2, \ldots N$ leads to

$$\begin{aligned} y(n+1) &= -a_0 y(1) - \ldots - a_{n-1} y(n) + b_0 u(1) + \ldots + b_m u(n+1) + \epsilon(n+1) \\ y(n+2) &= -a_0 y(2) - \ldots - a_{n-1} y(n+1) + b_0 u(2) + \ldots + b_m u(n+1) + \epsilon(n+2) \\ &\vdots \\ y(N) &= -a_0 y(N-n) - \ldots - a_{n-1} y(N+1) + b_0 u(N-n) + \ldots + b_m u(N) + \epsilon(N) \end{aligned}$$

Ordinary least square method

The obtained set of equations for N measures can be written in a matrix form as

$$\underbrace{\begin{pmatrix} y(n+1) \\ y(n+2) \\ \vdots \\ \\ \vdots \\ y(N) \end{pmatrix}}_{Y} = \underbrace{\begin{pmatrix} -y(1) & \dots & -y(n) & u(1) & \dots & u(n+1) \\ -y(2) & \dots & -y(n+1) & u(2) & \dots & u(n+2) \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ -y(N-n) & \dots & -y(N-1) & u(N-n) & \dots & u(N) \end{pmatrix}}_{\Phi} \underbrace{\begin{pmatrix} a_0 \\ a_1 \\ \vdots \\ a_{n-1} \\ b_0 \\ \vdots \\ b_n \end{pmatrix}}_{\theta} + \underbrace{\begin{pmatrix} \epsilon(n+1) \\ \epsilon(n+2) \\ \vdots \\ \\ \vdots \\ \epsilon(N) \end{pmatrix}}_{\epsilon}$$

Where the n first data are considered as initial conditions.

Ordinary least square method

To estimate θ, a quadratic criterion is selected. It is equal to the sum of squared errors $\epsilon(i)$:

$$\begin{aligned} J(\theta) &= \sum_{i=n+1}^{N} \epsilon^2(i) = \epsilon^\top \cdot \epsilon \\ &= (Y - \Phi\theta)^\top \cdot (Y - \Phi\theta) \\ &= (Y^\top - \theta^\top \Phi^\top) \cdot (Y - \Phi\theta) \\ &= Y^\top Y - Y^\top \Phi\theta - \theta^\top \Phi^\top Y + \theta^\top \Phi^\top \Phi\theta \end{aligned}$$

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$\Phi^\top\Phi$ is assumed to be invertible and $J(\theta)$ can then be written (as a sum of two terms with one which does not depend on θ) :

$$\begin{aligned} J(\theta) = & \left(\theta - (\Phi^\top\Phi)^{-1}\Phi^\top Y\right)^\top \Phi^\top\Phi \left(\theta - (\Phi^\top\Phi)^{-1}\Phi^\top Y\right) \\ & + Y^\top \left(\mathbf{I} - \Phi(\Phi^\top\Phi)^{-1}\Phi^\top\right) Y \end{aligned}$$

Ordinary least square method

$$\begin{aligned} J(\theta) &= \left(\theta - (\Phi^\top\Phi)^{-1}\Phi^\top Y\right)^\top \Phi^\top\Phi \left(\theta - (\Phi^\top\Phi)^{-1}\Phi^\top Y\right) \\ &\quad + Y^\top \left(\mathbf{I} - \Phi(\Phi^\top\Phi)^{-1}\Phi^\top\right) Y \end{aligned}$$

$J(\theta)$ is minimum if the term $\left(\theta - (\Phi^\top\Phi)^{-1}\Phi^\top Y\right)^\top \Phi^\top\Phi \left(\theta - (\Phi^\top\Phi)^{-1}\Phi^\top Y\right)^\top$ is equal to zero, that is

$$\boxed{\hat{\theta} = (\Phi^\top\Phi)^{-1}\Phi^\top Y}$$

Ordinary least square method

The implementation of the method can be broken down into the following steps :

1. Formulate the model as a "standard" difference equation

$$y_M(i+n) = -a_{n-1}y_M(i+n-1) - \ldots - a_0 y_M(i) + b_n u(i+n) + \ldots + b_0 u(i)$$

   A change of variable, if necessary, should make it possible that the oldest iterate of u is indexed by i.

2. Identify n.
3. Build matrices Y and Φ.

Ordinary least square method

4. Parameters are estimated using formula $\hat{\theta} = (\Phi^\top \Phi)^{-1} \Phi^\top Y$.
5. Criterion $J(\hat{\theta})$ is evaluated to validate or to reject the estimation.

Estimation using a random signal

On-line estimation procedure

Principle

Parameters θ are estimated iteratively using input and output measures acquired up to the current instant.

On-line estimation procedure

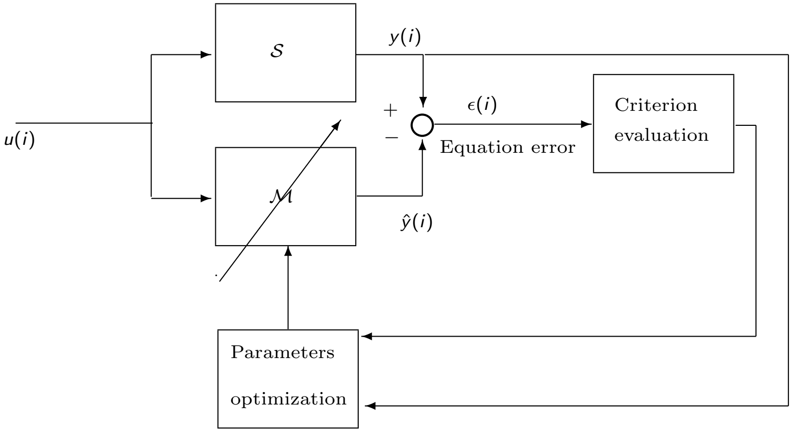

Figure: Principle of the on-line parameters estimation

On-line estimation procedure

Advantages

The estimation is then done "on the fly", and this approach is the only one to be valid if the identification

- is used for adaptative control;
- is used for time-varying processes (*non-stationary*).

$\rightsquigarrow$ *This method enables us to update the estimation of parameters along the process evolution.*

Problem

Having evaluated parameters vector $\hat{\theta}_j$ by means of j measures

$$\hat{\theta}_j = (\Phi_j^\top \Phi_j)^{-1} \Phi_j^\top Y_j$$

How to obtain, consecutively to a new measure $y(j+1)$,
the new vector $\hat{\theta}_{j+1}$?

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How to obtain, consecutively to a new measure $y(j+1)$, the new vector $\hat{\theta}_{j+1}$?

This iterative method is said to be **recursive** since it enables us to evaluate $\hat{\theta}_{j+1}$ without starting from scratch calculus $(\Phi_{j+1}^\top \Phi_{j+1})^{-1} \Phi_{j+1}^\top Y_{j+1}$ but by means of a calculus implying $\hat{\theta}_j$ and a correction taking into account additional measure $y(j+1)$.

Notations

$$\Phi_j = \begin{pmatrix} \varphi(n+1) \\ \varphi(n+2) \\ \vdots \\ \varphi(j) \end{pmatrix} = \begin{pmatrix} -y(1) & \dots & -y(n) & u(1) & \dots & u(n+1) \\ -y(2) & \dots & -y(n+1) & u(2) & \dots & u(n+2) \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ -y(j-n) & \dots & -y(j-1) & u(j-n) & \dots & u(j) \end{pmatrix}$$

$$\Phi_{j+1} = \begin{pmatrix} \Phi_j \\ \varphi(j+1) \end{pmatrix}$$

$$Y_j = \begin{pmatrix} y(n+1) \\ y(n+2) \\ \vdots \\ y(j) \end{pmatrix}$$

$$Y_{j+1} = \begin{pmatrix} Y_j \\ y(j+1) \end{pmatrix} .$$

Recursive least square method

Taking into account the $(j+1)$-th measure, $\hat{\theta}_{j+1}$ can be written:

$$\begin{aligned}
\hat{\theta}_{j+1} &= (\Phi_{j+1}^\top \Phi_{j+1})^{-1} \Phi_{j+1}^\top Y_{j+1} \\
&= \left(\begin{pmatrix} \Phi_j \\ \varphi(j+1) \end{pmatrix}^\top \begin{pmatrix} \Phi_j \\ \varphi(j+1) \end{pmatrix} \right)^{-1} \begin{pmatrix} \Phi_j \\ \varphi(j+1) \end{pmatrix}^\top \begin{pmatrix} Y_j \\ y(j+1) \end{pmatrix} \\
&= \left[\Phi_j^\top \Phi_j + \varphi^\top(j+1)\varphi(j+1) \right]^{-1} (\Phi_j^\top Y_j + \varphi^\top(j+1) y(j+1))
\end{aligned}$$

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The idea is to show the relation between $\hat{\theta}_{j+1}$ and $\hat{\theta}_j$ in this expression.

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The idea is to show the relation between $\hat{\theta}_{j+1}$ and $\hat{\theta}_j$ in this expression.

The algorithm

For each measure j :

$$P_{j+1} = P_j - P_j \varphi^\top(j+1) \left[\varphi(j+1) P_j \varphi^\top(j+1) + 1 \right]^{-1} \varphi(j+1) P_j$$

$$\hat{\theta}_{j+1} = \hat{\theta}_j + P_{j+1} \varphi^\top(j+1) \left[y(j+1) - \varphi(j+1)\hat{\theta}_j \right]$$

Remarks

$$P_{j+1} = P_j - P_j\varphi^\top(j+1)\left[\varphi(j+1)P_j\varphi^\top(j+1)+1\right]^{-1}\varphi(j+1)P_j$$

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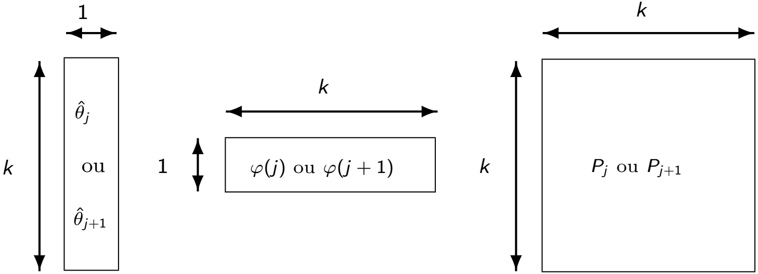
Remark 1



$\varphi(j+1) P_j \varphi^\top(j+1)$ is a scalar $\Rightarrow$ no matrix inversion is needed!

Remarks (ctd)

$$P_{j+1} = P_j - P_j\varphi^\top(j+1)\left[\varphi(j+1)P_j\varphi^\top(j+1)+1\right]^{-1}\varphi(j+1)P_j$$

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Remark 2

It should be clear that $\hat{\theta}_{j+1}$ is deduced from the preceding value $\hat{\theta}_j$ and from a correction term which takes into account the new measure.

Remarks (ctd)

$$P_{j+1} = P_j - P_j \varphi^\top(j+1) \left[\varphi(j+1) P_j \varphi^\top(j+1) + 1\right]^{-1} \varphi(j+1) P_j$$

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Remark 3

An *oversight factor* of the formest values can be added. It is denoted λ:

$$P_{j+1} = \frac{P_j - P_j\varphi^\top(j+1)\left[\varphi(j+1)P_j\varphi^\top(j+1)+\lambda\right]^{-1}\varphi(j+1)P_j}{\lambda}$$

The value $\lambda = 0.95$ makes it possible a fast oversight and, as a by-product, the pursuit of a strong instationarity.
Classically, $0.95 \leq \lambda \leq 0.99$.

Remarks (end)

$$P_{j+1} = P_j - P_j \varphi^\top(j+1) \left[\varphi(j+1) P_j \varphi^\top(j+1) + 1\right]^{-1} \varphi(j+1) P_j$$

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Remark 4

As any recurrence, it must be initialized (values of $\hat{\theta}_0$ and P_0). One shall consider:

$$\hat{\theta}_0 = \begin{pmatrix} 0 \\ \vdots \\ \vdots \\ 0 \end{pmatrix} \text{ and } P_0 = \alpha \begin{pmatrix} 1 & 0 & \dots & 0 \\ 0 & 1 & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \dots & 0 & 1 \end{pmatrix}$$

with α very large. If we have information on the value of $\hat{\theta}_0$, a lower value can be considered for α.