RELIABILITY BLOCK DIAGRAM (RBD)

1. System performance
2. RBD definition
3. Structure function
4. Reliability function of a non repairable complex system
5. Reliability metrics for a repairable complex system

Problem:

- Individual definition of the failure modes and components
- What about the global performance of the « system »?

Assumptions:

- $R_S(t) = h_S(t; x_1(t), x_2(t), \dots, x_n(t))$

  where

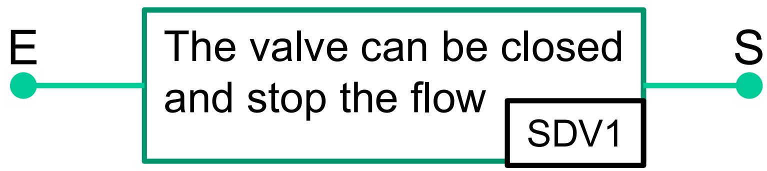
  - $x_i(t)$ is the state of performing the function i, assumed to be independent on the other states $x_j, j \neq i$
  - $h_s(t; \cdot)$ is the reliability function which is a function of the level of the realization of the set of the sytem functions and of the structure (architecture) of the system.

Reliability Block Diagram definition

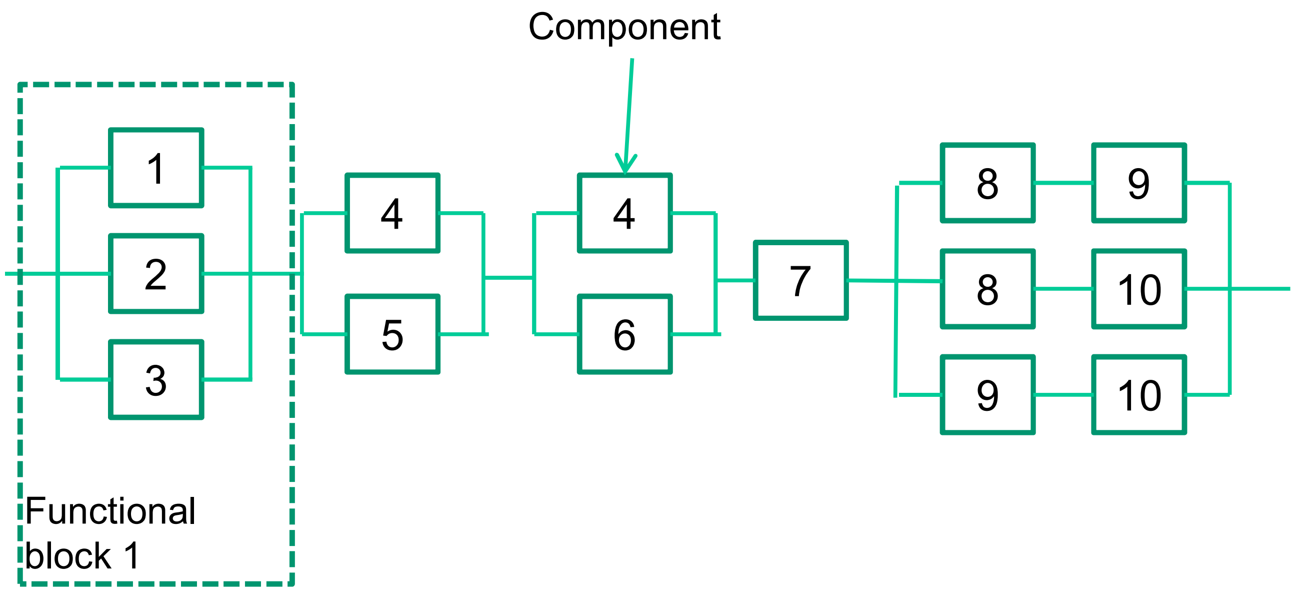
Definition

- A Reliability Block Diagram is a **success-oriented graph** which illustrates from a logical perspective how the different functional blocks ensure the global mission/function of the system. The structure of the reliability block diagram is mathematically described through **structure functions** which allow to assess some reliability measures of the (complex) system.

Function model



Decomposition of a complex system in functional blocks



The classical architectures

« Serie » structure

- A system which is operating if and only of all of its n components are operating is called a serie structure/system.

« Parallel » structure

- A system which is operating if at least one of its n components is operating is called a parallel structure/system.

« $k$oo$n$ » structure

- A system which is operating if at least k of its n component are operating is called a k over n structure/system

Example

Let consider 2 independent safety-related valves V1 and V2 which are physically installed in series.

In normal condition, the valves are opened and automatically close for shuting down the process in emergency case.

Questions :

- Identify the functions of the system
- Draw the reliability block diagram

Definition

Let consider a system with *n* independent components and let define its state vector $\boldsymbol{x} = (x_1, x_2, \dots, x_n)$ with:

$$x_i = \begin{cases} 1 & if\ the\ component\ i\ ok \\ 0 & if\ not \end{cases}$$

The binary function $\phi(\boldsymbol{x}) = \phi(x_1, x_2, \dots, x_n)$ defined by

$$\phi(\boldsymbol{x}) = \begin{cases} 1 & if\ the\ system\ is\ operational \\ 0 & if\ it\ is\ in\ the\ failed\ state \end{cases}$$

Is called **system structure function** or simply **structure**.

Write the structure function:

- For a serie system
- For a parallel system
- For a 2oo3 system

Definition

A component *i* is said **non-relevant** if $\phi(1_i, \boldsymbol{x}) = \phi(0_i, \boldsymbol{x})$ for all $(\cdot_i, \boldsymbol{x})$ where $(\cdot_i, \boldsymbol{x})$ is the vector for which the state of the component *i* is 1 or 0

A system is **coherent** if all of its components are relevant and its structure function is non-decreasing in its parameters

- **Theorems**
  - $\phi(\mathbf{0}) = 0$ and $\phi(\mathbf{1}) = 1$
  - $\prod_{i=1}^{n} x_i \leq \phi(\boldsymbol{x}) \leq \coprod_{i=1}^{n} \boldsymbol{x}_i$
  - $\phi(\boldsymbol{x} \sqcup \boldsymbol{y}) \geq \phi(\boldsymbol{x}) \sqcup \phi(\boldsymbol{y})$
  - $\phi(\boldsymbol{x} \cdot \boldsymbol{y}) \leq \phi(\boldsymbol{x}) \cdot \phi(\boldsymbol{y})$

Reliability function of a non repairable complex system

The state of the component i at time t is a random variable denoted $X_i(t)$

Let define:

- The reliability of component i: $r_i(t) = P(X_i(t) = 1) = E(X_i(t))$
- The reliability of system *S*

$$R_S(t) = P\big(\phi\big(\boldsymbol{X}(t)\big) = 1\big) = E\left(\phi\big(\boldsymbol{X}(t)\big)\right) = h(\boldsymbol{r}(t))$$

if independence on the components

Reliability function of a non repairable complex system

Serie system

Reliability assessment for a system with n independent components

$$R_S(t) = E\left(\phi(\boldsymbol{X}(t))\right) = E\left(\prod_{i=1}^{n} X_i(t)\right) = \prod_{i=1}^{n} E(X_i(t)) = \prod_{i=1}^{n} r_i(t)$$

$$= e^{-\int_0^t \boxed{\sum_{i=1}^{n} z_i(u)} du}$$

System failure rate function

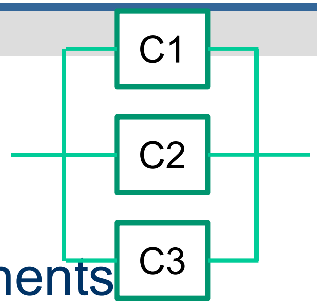
Let remark:

$$R_S(t) \leq \min_i (r_i(t))$$

Application:

What is the reliability of a serie system with n independent components when the failure rates are constant? What is the associated MTTF?

Reliability function of a non repairable complex system



Parallel system

Reliability assessment for a system with n independent components

$$R_S(t) = E\left(\phi\left(\boldsymbol{X}(t)\right)\right) = E\left(\coprod_{i=1}^{n} X_i(t)\right) = 1 - \prod_{i=1}^{n}\left(1 - E\left(X_i(t)\right)\right)$$

$$= 1 - \prod_{i=1}^{n}\left(1 - r_i(t)\right) = \coprod_{i=1}^{n} r_i(t)$$

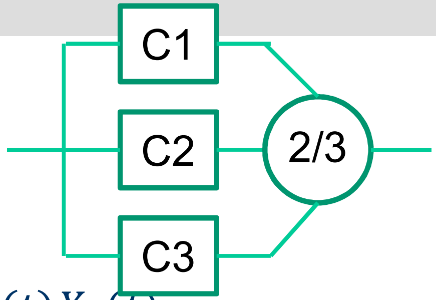
If the component failure rates are constant:

$$R_S(t) = 1 - \prod_{i=1}^{n}\left(1 - e^{-\lambda_i \cdot t}\right)$$

Application:

What is the reliability function, the MTTF and the failure rate of a parallel system with 2 independent components when $\lambda_i = \lambda$?

Reliability function of a non repairable complex system



2oo3 system

The system structure function

$$\phi(\boldsymbol{X}(t)) = X_1(t)X_2(t) + X_1(t)X_3(t) + X_2(t)X_3(t) - 2X_1(t)X_2(t)X_3(t)$$

The associated reliability function:

$$R_S(t) = R_1(t)R_2(t) + R_1(t)R_3(t) + R_2(t)R_3(t) - 2R_1(t)R_2(t)R_3(t)$$

If $z_i(t) = \lambda$,

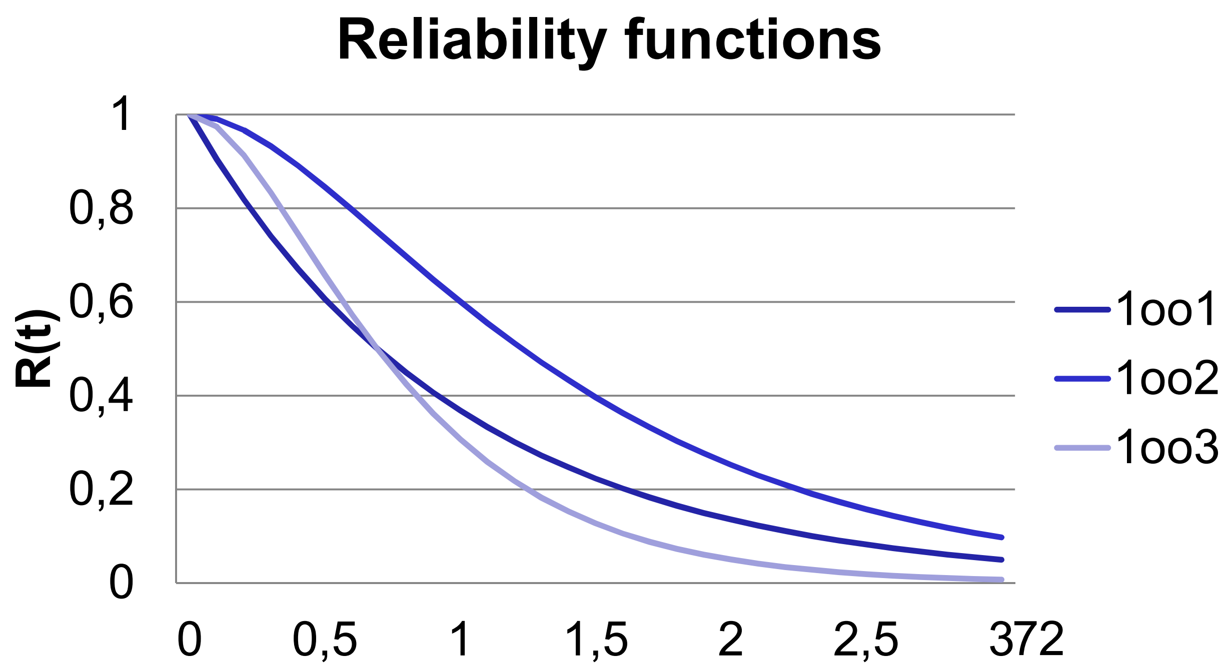
$$R_S(t) = 3e^{-2\lambda t} - 2e^{-3\lambda t}$$

$$z_S(t) = \frac{6\lambda\left(e^{-2\lambda t} - e^{-3\lambda t}\right)}{3e^{-2\lambda t} - 2e^{-3\lambda t}} \xrightarrow{t\to\infty} 2\lambda$$

$$MTTF = \int_0^\infty R_S(t)dt = \frac{5}{6}\frac{1}{\lambda}$$

Reliability function of a non repairable complex system

Structure	Fonction de fiabilité	MTTF
1oo1	$e^{-\lambda t}$	$\frac{1}{\lambda}$
1oo2	$2e^{-\lambda} - e^{-2\lambda t}$	$\frac{3}{2}\frac{1}{\lambda}$
2oo3	$3e^{-2\lambda t} - 2e^{-3\lambda t}$	$\frac{5}{6}\frac{1}{\lambda}$



Redundancies

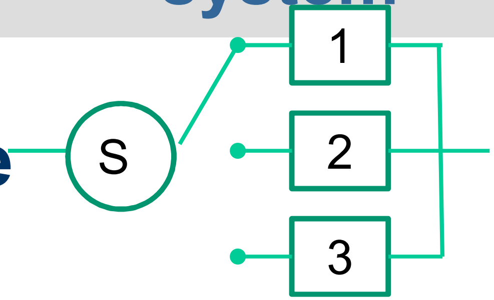
The reliabilitlty improvement for a critical functionn is:

- The search in the improvement in the intrinsic reliability of the function (choice of very high reliable components)
- The introduction of **active** or **passive redundancies**.

It the components are working in parallel (in the same time), we call active redundancy. The components share the function load as far as one of the components fails.

If one component is in standby for ensuring the function as soon as the other component fails, we call passive redundancy.

- It the stanby component is not involved in the another function and that we can assume no failure of this component in this interval, we talk about **cold standby**.
- If the standby component ensures a reduced working load, we said **partly loaded redundancy**.



Performance analysis of the passive redundancies

Passive redundancy, perfect replacement, no repair

Cold passive redundancy, imperfect replacement, no repair

Partly load passive redundancy, imperfect replacement, no repair

Reliability optimization

The objective in the product design can be to maximize the reliability on a given time horizon. This maximization could be done with a budget constrainst or at the lower cost. Other criteria could be associated (minimization of the weight) or some extra constraints (volume, design, …).

This optimization consists in finding the best compromise between the reliability improvement and the other criterion. Multi-objective optimization technics (Operations Research) can be used to solve such a problem

$$\max_{n \in \mathbb{N}^k} f^i(n), i \in \{1, \dots, m\}$$

Reliability function of a non repairable complex system

Problem 1 :

- We search the optimal structure of a serie system which has to perform *k* sub-functions. The problem is to maximize the reliability on a fixed interval t with a max investisment budget $Cmax$.
- Assumption:
  - Each of the subfunctions can be performed by a specific component at a unitary cost c_k. Let consider the failure rates be constant λ_k.

This is equivalent to find the vector $\boldsymbol{n}^* = (n_1{}^*, n_k{}^*, \dots, n_k{}^*)$ such as

$$R(\boldsymbol{n}^*, t) = \max_{\boldsymbol{n} \in \mathbb{N}^k} \left\{ R(\boldsymbol{n}, t) \middle| \sum_{i=1}^{k} n_i c_i \leq C_{max} \right\}$$

with $\mathrm{R}(\boldsymbol{n}, t) = \prod_{i=1}^{k} \left[1 - (1 - e^{-\lambda_i . t})^{n_i} \right]$

Problem 2 :

- Find an optimal structure of a serie system for performing a set of k sub-functions. We search to maximize the reliability on a fixed horizon t at a minimal cost.
- Assumption:
  - Every function is performed by a specific component with a unit cost c_k and an operating cost per unit of time g_k. Let assume that the number of components is fixed and respective constant failure rates λ_k.

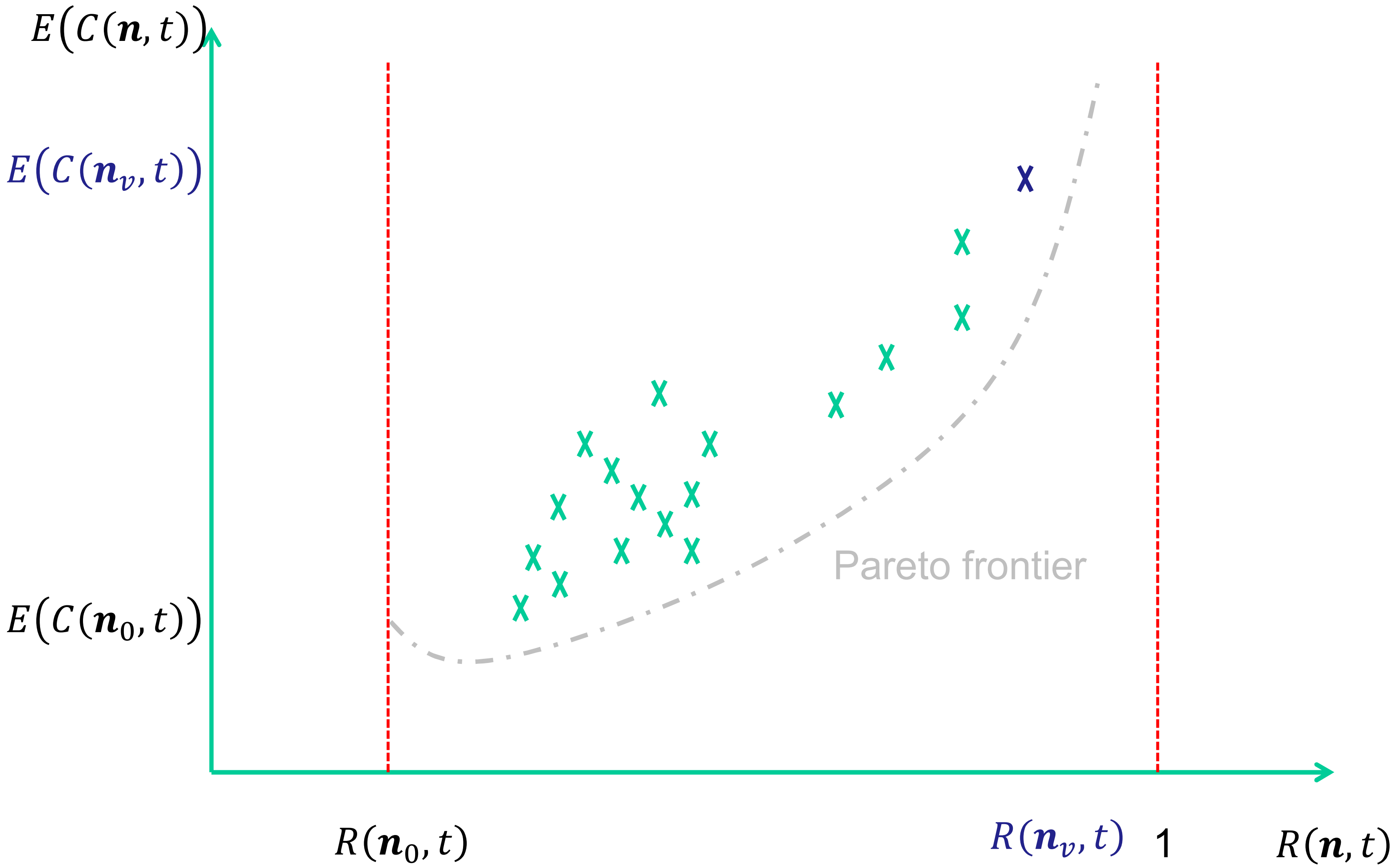
Heuristic for the reliability maximization and cost minization:

- Step 0 : Elaborate the objective functions for the initial system structure:

$$\left\{R(\boldsymbol{n}_0,t)=e^{-\sum_{i=1}^{k}\lambda_i.t}, E\big(C(\boldsymbol{n}_0,t)\big)=\sum_{i=1}^{k}[c_i+g_i\cdot E(\min(T_S,t))]\right\}$$

- Step *v*
  - Add a component at the least reliable block a ; $n_a{}^v = n_a{}^{v-1}+1$ and $n_i{}^v = n_i{}^{v-1}, i\neq a$
  - Evaluate the reliability function of the block a, $R_a\big(n_a{}^v, t\big)=1-(1-e^{-\lambda_a.t})^{n_a}$
  - Evaluate the system reliability $R(\boldsymbol{n}_v,t)=\prod_{i\neq a}R_i(n_i{}^{v-1},t)\cdot R_a(n_a{}^v,t)$
  - Estimate the mean time of failure before : $E\big(\min(T_S{}^v,t)\big)=\int_0^t R(\boldsymbol{n}_v,u)du$
  - Estimate the average cost $E\big(C(\boldsymbol{n}_v,t)\big)=\sum_{i=1}^{k}\left[n_i{}^v\ \Big(c_i+g_i\cdot E\big(\min(T_S{}^v,t)\big)\Big)_i\right]$
  - Dessiner $\{R(\boldsymbol{n}_v,t), E\big(C(\boldsymbol{n}_v,t)\big)\}$

Master ISMP - Castanier



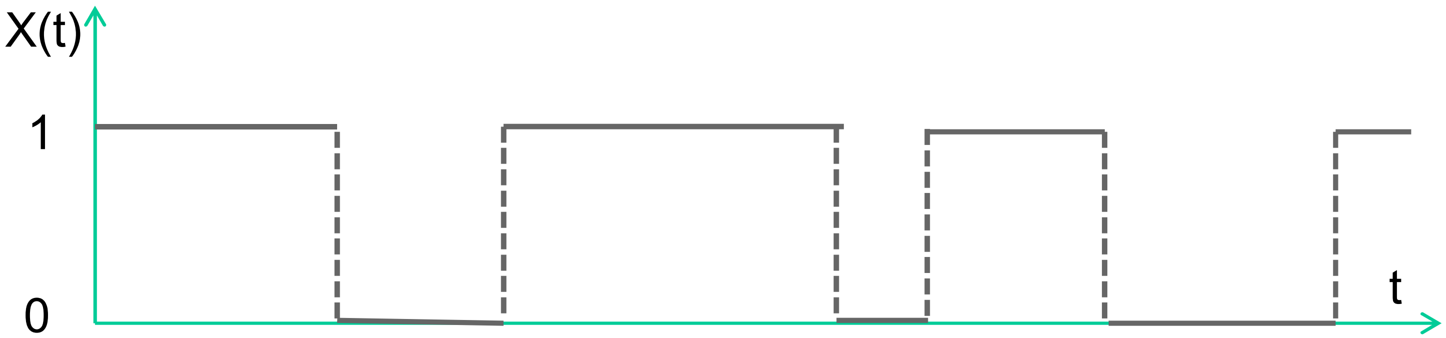
$E(C(\boldsymbol{n},t))$
$E(C(\boldsymbol{n}_v,t))$
$E(C(\boldsymbol{n}_0,t))$
Pareto frontier
$R(\boldsymbol{n}_0,t)$
$R(\boldsymbol{n}_v,t)$
1
$R(\boldsymbol{n},t)$

Analyzis of repairable systems

A repairable system is a system subject to degradation and be in a better state after a repair.

Example :

- For a simple system



- What are the different questions in case of repairable systems?
  - What is the probability that the system is operating at a given date?
  - What is the average operation time on a given time horizon?
  - What is the time percentage the system is unavailable?

Availability of a single unit

(λ, μ)

At time *t* :

- $p_0(t)$: probability of being in the good state ($p_0(t=0)=1$)
- $p_1(t)$: probability of being in the failed state ($p_1(t=0)=0$)

To ensure the operation at time $t+\delta t$ ($p_0(t+\delta t)$), it needs:

- The system is operating at *t* and there is no failure in $(t, t+\delta t)$
- The system is on repair at *t* and be operational at $(t, t+\delta t)$

Then

$$p_0(t+\delta t) = p_0(t)\cdot(1-\lambda\cdot\delta t) + \big(1-p_0(t)\big)\cdot\mu\cdot\delta t$$

- By dividing by δt and if δt tends to 0,

$$p_0'(t) + (\lambda+\mu)p_0(t) = \mu$$

- First-order differential equation with the following solution

$$p_0(t) = \frac{1}{\lambda+\mu}\left(\mu+\lambda e^{-(\lambda+\mu)t}\right)$$

So,

- The (instantaneous) availability at time *t*:

$$A(t) = \frac{1}{\lambda + \mu}\left(\mu + \lambda e^{-(\lambda+\mu)t}\right)$$

- The asymptotic availability $(t \to \infty)$ or « mean » or « average »:

$$A_\infty = \frac{\mu}{\lambda + \mu} = \frac{MTTF}{MTTF + MTTR}$$

- The average availabilty on an interval (t_1, t_2)

$$A_{av}(t_1, t_2) = \frac{1}{t_2 - t_1}\int_{t_1}^{t_2} A(t)dt$$

- The average reliability from the fiLa disponibilité moyenne depuis la mise en service

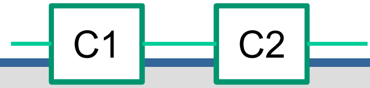
$$A_{av}(\tau) = \frac{1}{\tau}\int_0^{\tau} A(t)dt$$

- La disponibilité moyenne asymptotique

$$A_{av} = \lim_{\tau\to\infty} A_{av}(\tau) = \lim_{\tau\to\infty}\frac{1}{\tau}\int_0^{\tau} A(t)dt$$

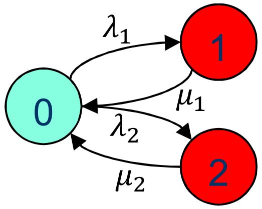
- Moreover,
  - The asymptotic unavailability is $\bar{A}_\infty = 1 - A_\infty = \frac{\lambda.MTTR}{1+\lambda.MTTR} \approx \lambda.MTTR$
  - The expected number of failures on an interval : $W(t) \approx \frac{t}{MTTF+MTT}$
  - The operational unavailability: $\bar{A}_{OP} = 1 - A_{OP} = \frac{Mean\ planned\ maintenance\ duration + Mean\ unplanned\ maintenance\ duration}{Mission\ time}$

Availability of a serie system



$(\lambda_1, \mu_1)\ (\lambda_2, \mu_2)$

Markov graph



Assumption: $\lambda = \lambda_1 = \lambda_2$ et $\mu = \mu_1 = \mu_2$

One-step transition matrix $M = \begin{bmatrix} -2\lambda & \lambda & \lambda \\ \mu & -\mu & 0 \\ \mu & 0 & -\mu \end{bmatrix}$ to solve $\frac{dP}{dx}(t) = P(t)M$ by Laplace transform.

The Laplace transformed is the solution of:

$$(\mathcal{L}_1(p), \mathcal{L}_2(p), \mathcal{L}_3(p)) = (1,0,0)(pI - M)^{-1} = (1,0,0)\begin{bmatrix} p+2\lambda & -\lambda & -\lambda \\ -\mu & p+\mu & 0 \\ -\mu & 0 & p+\mu \end{bmatrix}^{-1}$$

The availability is given by the sum of the operating states given the initial condition

$$\mathcal{L}A_S(p) = (1,0,0)\begin{bmatrix} p+2\lambda & -\lambda & -\lambda \\ -\mu & p+\mu & 0 \\ -\mu & 0 & p+\mu \end{bmatrix}^{-1}\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \frac{\widetilde{M}_{11}}{\Delta_M}$$

Nota: This is equivalent to a 2-state transition graph and we directly have

$$\boxed{A_S(t) = \frac{\mu}{2\lambda + \mu} + \frac{2\lambda}{2\lambda + \mu} e^{-(2\lambda + \mu)t}}$$

Availability of a serie system

In the general case, we can prove:

$$MTTR_S = \sum_{i=1}^{n} \frac{\lambda_i}{\sum_{i=1}^{n} \lambda_i} MTTR_i = \frac{\sum_{i=1}^{n} \frac{\lambda_i}{\mu_i}}{\sum_{i=1}^{n} \lambda_i}$$

So, the system repair rate is

$$\mu_S = \frac{1}{MTTR_S} = \frac{\sum_{i=1}^{n} \lambda_i}{\sum_{i=1}^{n} \frac{\lambda_i}{\mu_i}}$$

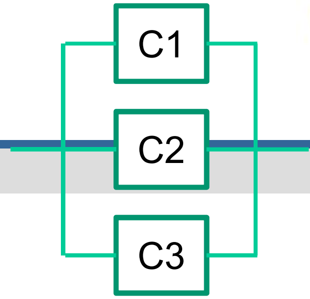
And, because the system failure rate is $\lambda_s = \sum_{i=1}^{n} \lambda_i$ and the asymptotic availability $A_S = \frac{MTTF_S}{MTTF_S+MTTR_S} = \frac{\mu_S}{\lambda_S+\mu_S}$, we have

$$A_S = \frac{1}{\sum_{i=1}^{n} \frac{1}{A_i} - n + 1}$$

Nota: **If A_i is closed to one**, by the structure function approach, we can approximate:

$A_S(t) = E\big(\phi(\boldsymbol{X}(t))\big) \cong \prod_{i=1}^{n} A_i(t) = \prod_{i=1}^{n} \left[\frac{1}{\lambda_i+\mu_i}\left(\mu_i+\lambda_i e^{-(\lambda_i+\mu_i)t}\right)\right]$ and $A_S = \prod_{i=1}^{n} \left[\frac{\mu_i}{\lambda_i+\mu_i}\right]$

Availability of a parallel system



(λ_i, μ_i)

The availability is

Only if independance

$$A_S(t) = E\left(\phi(\boldsymbol{X}(t))\right) = 1 - \prod_{i=1}^{n} [1 - A_i(t)]$$

$$= 1 - \prod_{i=1}^{n} \left[\frac{1}{\lambda_i + \mu_i}\left(\lambda_i + \mu_i e^{-(\lambda_i + \mu_i)t}\right)\right]$$

And so, the asymptotic availibility can be approximated

$$A_S = 1 - \prod_{i=1}^{n} \left[\frac{\lambda_i}{\lambda_i + \mu_i}\right]$$

Résolution du cas à 2 redondances actives identiques et un réparateur

Markov transition graph where Ei is the event « i components are operating »

One-step transition matrix $M = \begin{bmatrix} -2\lambda & 2\lambda & 0 \\ \mu & -(\lambda+\mu) & \lambda \\ 0 & \mu & -\mu \end{bmatrix}$ to solve $\frac{dP}{dx}(t) = P(t)M$ with Laplace.

The Laplace transformed is the solution of

$$(\mathcal{L}_1(p), \mathcal{L}_2(p), \mathcal{L}_3(p)) = (1,0,0)(pI - M)^{-1} = (1,0,0)\begin{bmatrix} p+2\lambda & -2\lambda & 0 \\ \mu & p+\lambda+\mu & -\lambda \\ 0 & -\mu & p+\mu \end{bmatrix}^{-1}$$

And because the availability is the sum of the operating state

$$\mathcal{L}A_S(p) = (1,0,0)\begin{bmatrix} p+2\lambda & -2\lambda & 0 \\ \mu & p+\lambda+\mu & -\lambda \\ 0 & -\mu & p+\mu \end{bmatrix}^{-1}\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} = \frac{\widetilde{M}_{11} + \widetilde{M}_{21}}{\Delta_M}$$

With the matrix determinant, we have

$$\Delta_M = p(p^2 + p(2\mu + 3\lambda) + \mu^2 + 2\lambda^2 + 2\lambda\mu) = p(p - S_1)(p - S_2)$$

$$\mathcal{L}A_S(p) = \frac{p^2 + p(2\mu + 3\lambda) + \mu^2 + 2\lambda\mu}{p(p - S_1)(p - S_2)} = \frac{A}{p} + \frac{B}{p - S_1} + \frac{C}{p - S_2}$$

With $S_1 = -\frac{1}{2}(3\lambda + 2\mu + \sqrt{\lambda^2 + 4\lambda\mu})$; $S_2 = -\frac{1}{2}(3\lambda + 2\mu - \sqrt{\lambda^2 + 4\lambda\mu})$ and $\begin{cases} AS_1S_2 = \mu^2 + 2\lambda\mu \\ A + B + C = 1 \\ -(A+B)S_2 - (A+C)S_1 = 2\mu + 3\lambda \end{cases}$

And finally we can prove

$$A_S(t) = \frac{\mu^2 + 2\lambda\mu}{\mu^2 + 2\lambda\mu + 2\lambda^2} - \frac{2\lambda(S_2 e^{S_1 t} - S_1 e^{S_2 t})}{S_1 S_2 (S_1 - S_2)}$$

Optimizing a structure

The structure optimization is in this context to define:

- The nature of the redundancy
- The number of redundant components/functions
- The number of repairmen

The problem is a complex discrete optimization problem especially because of the dependencies due to the repairmen availabilities.

It the structure is complex, an optimization method based on stochastic simulation (MCMC or Generalized SPNN) could be used.

We can also used the previous approx if the approximation quality is relevant (low repair time comparing to operating time)